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ON THE EXTENDED JIANG SUBGROUP OF THE FUNDAMENTAL GROUP

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Abstract. We introduce an extended Jiang subgroup $J(f, x_0, G)$ of the fundamental group of a transformation group as a generalization of the Jiang subgroup $J(f, x_0)$ and show some properties of this extended Jiang subgroup.

1. Introduction

F. Rhodes [4] introduced the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G) as a generalization of the fundamental group of a topological space X and showed that $\sigma(X, x_0, G)$ is isomorphic to $\pi_1(X, x_0) \times G$ if (G, G) admits a family of preferred paths at e. B. J. Jiang [3] introduced the Jiang subgroup $J(f, x_0)$ of the fundamental group of a topological space X.

In the same line with D. H. Gottlieb [1], Jiang Bo-Ju [3] defined the trace group $J(f, x_0)$ of cyclic homotopy from a continuous self-map f to f which is also a subgroup of a fundamental group. The Jiang's subgroup $J(f, x_0)$ is very important and interesting in fixed point theory.

In this paper we introduce an evaluation subgroup $J(f, x_0, G)$ of $\sigma(X, f(x_0), G)$ which is a generalization of $E(X, x_0, G)$ and $J(f, x_0)$ where f is a self-map from X to X. If $f = 1_X$, then $J(f, x_0, G) =$

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$E(X, x_0, G)$ which is defined in [6] and if $G = \{1_X\}$, then $J(f, x_0, G) = J(f, x_0)$ where is a Jiang's subgroup in [3].

2. Preliminaries and main results

Let (X, G, π) be a transformation group, where X is a path connected space with x_0 as base point. Given any element g of G, a path f of order g with base point x_0 is a continuous map $f : I \to X$ such that $f(0) = x_0$ and $f(1) = gx_0$. A path f_1 of order g_1 and a path f_2 of order g_2 give rise to a path $f_1 + g_1 f_2$ of order $g_1 g_2$ defined by the equations

$$(f_1 + g_1 f_2)(s) = \begin{cases} f_1(2s), & 0 \leq s \leq \frac{1}{2} \\ g_1 f_2(2s-1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Two paths f and f' of the same order g are said to homotopic if there is a continuous map $F : I^2 \to X$ such that

$$\begin{aligned} F(s,0) &= f(s), \ 0 \leq s \leq 1, \\ F(s,1) &= f'(s), \ 0 \leq s \leq 1, \\ F(0,t) &= x_0, \ 0 \leq t \leq 1, \\ F(1,t) &= gx_0, \ 0 \leq t \leq 1. \end{aligned}$$

The homotopy class of a path f of order g was denoted by $[f : g]$. Two homotopy classes of paths of different orders g_1 and g_2 are distinct, even if $g_1 x_0 = g_2 x_0$. F.Rhodes showed that the set of homotopy classes of paths of prescribed order with the rule of composition $*$ is a group, where $*$ is defined by $[f_1 : g_1] * [f_2 : g_2] = [f_1 + g_1 f_2 : g_1 g_2]$. This group was denoted by $\sigma(X, x_0, G)$, and was called the fundamental group of (X, G) with base point x_0.

Let f be a self-map of X. A homotopy $H : X \times I \to X$ is called an f-cyclic homotopy [3] if $H(x,0) = H(x,1) = f(x)$. This concept of a topological space is generalized to that of a transformation group. A continuous map $H : X \times I \to X$ is called an f-homotopy of order g if $H(x,0) = f(x), H(x,1) = gf(x)$, where g is an element of G. If H is an f-homotopy of order g, then the path $\alpha : I \to X$ given by $\alpha(t) = H(x_0, t)$ will be called the trace of H.

The trace subgroup of f-homotopies of prescribed order is defined by

$$J(f, x_0, g) = \{[\alpha : g] \in \sigma(x, f(x_0), G) | \exists \ f\text{-homotopy of order } g \text{ with trace } \alpha\}.$$

$J(1_X, x_0, G)$ was defined by $E(X, x_0, G)$ in [6] and $J(f, x_0, \{e\})$ was also defined by $J(f, x_0)$ in [3]. From this fact, we say that $J(f, x_0, G)$ is an extended Jiang subgroup.

It is easy to show that an extended Jiang subgroup $J(f, x_0, G)$ is a subgroup of $\sigma(X, f(x_0), G)$.

Let (X, G) be a transformation group and X^X be the space of all continuous mappings from X to X with compact-open topology. Let G act on X^X continuously by $\pi'(f, g) = gf$. Then (X^X, G, π') is a transformation group.

Let $P : X^X \to X$ be the evaluation map given by $P(f) = f(x_0)$. If X is a locally compact, then the evaluation map P is continuous. Since $P(gf) = gf(x_0) = gP(f)$, where $g \in G$ and $f \in X^X, (P, 1_G) : (X^X, G) \to (X, G)$ is a category mapping. Thus we know that $P_* : \sigma(X^X, 1_X, G) \to \sigma(X, x_0, G)$ defined by $P_*[\alpha : g] = [P \circ \alpha : g]$ is a homomorphism.

There is a natural homeomorphism $\phi : (X^X)^I \to X^{X \times I}$ given by $\phi(f)(x, s) = f(s)(x)$ for $x \in X$ and $s \in I$.

Note that $f \sim f'$ if and only if $\phi(f) \sim \phi(f')$. Motivated by the following theorem, we can consider $J(f, x_0, G)$ as a generalized evaluation subgroup of the fundamental group of a transformation group (X, G).

THEOREM A. *Let X be a pathwise connected CW-complex. Then*

$$P_* \sigma(X^X, f, G) = J(f, x_0, G).$$

The Jiang's result [3] can be generalized as follows.

THEOREM B. *Let f and k be self maps of X.*

(1) *$J(k, f(x_0), G) \subset J(k \circ f, x_0, G)$.*
(2) *If k is a homomorphism of (X, G), i.e., $kg(x) = gk(x)$ for any element g of G, then $k_\pi(J(f, x_0, G)) \subset J(k \circ f, x_0, G)$ where $k_\pi[\alpha : g] = [k\alpha : g]$ for any element $[\alpha : g]$ of $J(f, x_0, G)$.*

In [4], F. Rhodes showed that if λ is a path from x_0 to x_1, then λ induces an isomorphism $\lambda_* : \sigma(X, x_0, G) \to \sigma(X, x_1, G)$ such that $\lambda_*[\alpha : g] = [\lambda\rho + \alpha + g\lambda : g]$.

THEOREM C. *Assumes that X is a pathwise connected CW-complex. Let (X, G) be a transformation group. If λ is a path from x_0 to x_1 in X, then the induced homomorphism $(f\lambda)_*$ carries $J(f, x_0, G)$ isomorphically onto $J(f, x_1, G)$.*

THEOREM D. *If $f, k : X \to X$ are homotopic, then $J(f, x_0, G)$ and $J(k, x_0, G)$ are isomorphic.*

THEOREM E. *If $f : (X, G) \to (X, G)$ is a homomorphism, i.e., $fg(x) = gf(x)$ for any element g of G and x_1 belongs to g_0X_0 for some $g_0 \in G$, where X_0 is the path connected component of x_0, then $J(f, x_0, G)$ and $J(f, x_1, G)$ are isomorphic.*

THEOREM 1. *If $f, k : X \to X$ are homeomorphisms and $f(x_0) = k(x_0)$, then $J(f, x_0, G)$ is equal to $J(k, x_0, G)$.*

Proof. Let $[\alpha : g]$ be any element of $J(f, x_0, G)$. Then there exists a homotopy $H : X \times I \to X$ such that $H(x, 0) = f(x), H(x, 1) = gf(x)$ and $H(x_0, t) = \alpha(t)$. Let $K : X \times I$ be a homotopy such that $K = H \circ (f^{-1}k \times 1_t)$. So,

$$K(x, 0) = H(f^{-1}k(x), 0) = ff^{-1}k(x) = k(x)$$
$$K(x, 1) = H(f^{-1}k(x), 1) = gff^{-1}k(x) = gk(x)$$

and

$$\begin{aligned} K(x_0, t) &= H(f^{-1}k(x_0), t) = H(f^{-1}f(x_0), t) \\ &= H(x_0, t) = \alpha(t). \end{aligned}$$

Therefore $[\alpha : g]$ belongs to $J(k, x_0, G)$ and similarly $J(k, x_0, G)$ is contained in $J(f, x_0, G)$. Thus $J(f, x_0, G)$ is equal to $J(k, x_0, G)$. □

COROLLARY 2.

(1) *If $f, k : X \to X$ are homeomorphisms and $f(x_0) = k(x_0)$, then $J(f, x_0)$ is equal to $J(k, x_0)$.*
(2) *If $f : X \to X$ is a homeomorphism and $f(x_0) = x_0$, then $J(f, x_0, G)$ is equal to $E(X, x_0, G)$.*

THEOREM 3. *Suppose X, Y are pathwise connected CW-complexes respectively and two transformation group $(X, G), (Y, H)$ are the same homotopy type. Then $J(f, x_0, G)$ and $J(k, y_0, H)$ are isomorphic with $y_0 = \phi(x_0)$ where $\phi : X \to Y$ is a continuous function, and f, k are homeomorphisms.*

Proof. In [6], $E(X, x_0, G)$ and $E(Y, y_0, H)$ are isomorphic with $y_0 = \phi(x_0)$. $J(f, x_0, G)$ and $J(k, y_0, H)$ have the following diagram:

$$\begin{array}{ccc}
J(f, x_0, G) & \longrightarrow & J(k, y_0, H) \\
{\scriptstyle \phi_1}\big\downarrow & & \big\downarrow \\
E(X, x_0, f^{-1}Gf) & & E(Y, y_0, k^{-1}Hk) \\
{\scriptstyle \phi_3}\big\uparrow & & \big\uparrow \\
E(X, f(x_0), G) & & E(Y, k(y_0), H) \\
{\scriptstyle \phi_2}\big\uparrow & & \big\uparrow \\
E(X, x_0, G) & \longrightarrow & E(Y, y_0, H)
\end{array}$$

We prove the following properties.

(1) Let $\phi_1 : J(f, x_0, G) \to E(X, x_0, f^{-1}Gf)$ be the map such that $\phi_1[\alpha : g] = [f^{-1}\alpha : f^{-1}gf]$. Let $[\alpha : g]$ be any element of $J(f, x_0, G)$. Then there exists homotopy $H : X \times I \to X$ such that $H(x, 0) = f(x), H(x, 1) = gf(x)$ and $H(x_0, t) = \alpha(t)$. Therefore, there exists homotopy $H' : X \times I \to X$ such that $H'(x, t) = f^{-1}H(x, t)$. So, $H'(x, 0) = f^{-1}f(x) = x, H'(x, 1) = f^{-1}gf(x)$ and $H'(x_0, t) = f^{-1}H(x_0, t) = f^{-1}\alpha(t)$. In other words, ϕ_1 is well-defined since α is homotopic with β implies $f^{-1}\alpha$ is homotopic with $f^{-1}\beta$. Indeed , ϕ_1 is one to one and onto.

On the other hand, ϕ_1 is homomorphism since

$$\begin{aligned}
\phi_1([\alpha_1 : g_1] * [\alpha_2 g_2]) &= \phi_1[\alpha_1 + g_1\alpha_2 : g_1g_2] \\
&= [f^{-1}(\alpha_1 + g_1\alpha_2) : f^{-1}g_1g_2f] \\
&= [f^{-1}\alpha_1 + f^{-1}g_1\alpha_2 : f^{-1}g_1ff^{-1}g_2f] \\
&= [f^{-1}\alpha_1 : f^{-1}g_1f] * [f^{-1}\alpha_2 : f^{-1}g_2f] \\
&= \phi_1[\alpha_1 : g_1] * \phi_1[\alpha_2 : g_2].
\end{aligned}$$

Therefore, ϕ_1 is isomorphism.

(2) Let $\phi_2 : E(X, x_0, G) \to E(X, f(x_0), G)$ be the map defined in [6] . Since there exists a path λ from x_0 to $f(x_0), E(X, x_0, G)$ and $E(X, f(x_0), G)$ are isomorphic by [6].

(3) Let $\phi_3 : E(X, f(x_0), G) \to E(X, x_0, f^{-1}Gf)$ be the map such that $\phi_3[\alpha : g] = [f^{-1}\alpha : f^{-1}gf]$, let $[\alpha : g]$ be any element of $E(X, f(x_0), G)$. Then, there exists homotopy $H : X \times I \to X$ such that $H(f(x), 0) = f(x), H(f(x), 1) = gf(x)$ and $H(f(x_0), t) = \alpha(t)$. Therefore, there exists homotopy $H' : X \times I \to X$ such that $H'(x, t) = f^{-1} \circ H \circ (f \times 1_t)(x, t)$. So,

$$H'(x, 0) = f^{-1}H(f(x), 0) = f^{-1}f(x) = x,$$
$$H'(x, 1) = f^{-1}H(f(x), 1) = f^{-1}gf(x)$$

and

$$H'(x_0, t) = f^{-1}H(f(x_0), t) = f^{-1}\alpha(t)$$

In other words, ϕ_3 is well-defined since α is homotopic with β implies $f^{-1}\alpha$ is homotopic with $f^{-1}\beta$. Thus, ϕ_3 is isomorphism as ϕ_1. By (1), (2), (3), $J(f, x_0, G)$ and $E(X, x_0, G)$ are isomorphic and similarly $J(k, y_0, H)$ and $E(Y, y_0, H)$ are isomorphic. By [6], $J(f, x_0, G)$ and $J(k, y_0, H)$ are isomorphic with $y_0 = \phi(x_0)$. □

In [4], a transformation group (X, G) is said to admit a family K of preferred paths at x_0 if it is possible to associate with every element g of H a path k_g from gx_0 to x_0 such that the path k_e associated with 0identity element e of G is $\hat{x}_0$ which is the constant map such that $\hat{x}_0(t) = x_0$ for each $t \in I$ and for every pair of elements g, h, the path k_{gh} from ghx_0 to x_0 is homotopic to $gk_h + k_g$.

Definition 1. A family K of preferred paths at $f(x_0)$ is called a family of preferred f-traces at x_0 if for every preferred path k_g in $K, k_g\rho$ is the trace of f-homotopy of order g.

Theorem F. *Let (X, G, π) be a transformation group. If (G, G) admits a family of preferred paths at e, then (X, G) admits a family of preferred f-traces at x_0 for any self-map f of X.*

THEOREM G. *A transformation group (X, G) admits a family of preferred f-traces at x_0 if and only if $J(f, x_0, G)$ is a split extension of $J(f, x_0)$ by G.*

THEOREM H. *Let $f : X \to X$ be a homeomorphism. A transformation group (X, G) admits a family of preferred f-traces at x_0 if and only if there exists an isomorphism $\phi : J(f, x_0, G) \to J(f, x_0) \times G$ such that the diagram commutes*

$$\begin{array}{ccccccccc} O & \longrightarrow & J(f,x_0) & \longrightarrow & J(f,x_0,G) & \longrightarrow & G & \longrightarrow & O \\ & & \| & & \downarrow{\scriptstyle\phi} & & & & \| \\ O & \longrightarrow & J(f,x_0) & \longrightarrow & J(f,x_0)\times G & \longrightarrow & G & \longrightarrow & O \end{array}$$

We show that the existence of family of preferred f-traces on a transformation group does not depend on base point.

THEOREM 4. *Let (X, G) be a transformation group. If λ is a path from x_0 to x_1, then a family of preferred f-traces at x_0 gives rise to a family of preferred f-traces at x_1.*

Proof. Let $K = \{k_g | g \in G\}$ be a family of preferred f-traces at x_0. For each element g of G, let h_g be equal to $gf\lambda\rho + k_g + f\lambda$. We show that $H = \{h_g | g \in G\}$ is a family of preferred f-traces at x_1 since $h_e = f\lambda\rho + k_e + f\lambda \sim f(x_1)$ and

$$\begin{aligned} h_{g_1 g_2} &= (g_1 g_2) f\lambda\rho + k_{g_1 g_2} + f\lambda \\ &\sim (g_1 g_2) f\lambda\rho + g_1 k_{g_2} + k_{g_1} + f\lambda \\ &\sim (g_1 g_2) f\lambda\rho + g_1 k_{g_2} + g_1 f\lambda + g_1 f\lambda\rho + k_{g_1} + f\lambda \\ &\sim g_1(g_2 f\lambda\rho + k_{g_2} + f\lambda) + (g_1 f\lambda\rho + k_{g_1} + f\lambda) \\ &\sim g_1 h_{g_2} + h_{g_1}. \end{aligned}$$

Since the induced isomorphism $(f\lambda)_*$ carries $J(f, x_0, G)$ isomorphically onto $J(f, x_1, G)$ by Theorem C, $(f\lambda)_*[k_g\rho : g] = [f\lambda\rho + k_g\rho + gf\lambda : g] = [h_g\rho : g]$ belongs to $J(f, x_1, G)$ for any element $[k_g\rho : g]$ of $J(f, x_0, G)$. Thus $H = \{h_g | g \in G\}$ is a family of preferred f-traces at x_1.

The representation is natural with respect to change of base point in the sense that the following diagrams are commutative.

$$\begin{array}{ccc} J(f,x_0,G) & \xrightarrow{(f\lambda)_*} & J(f,x_1,G) \\ \phi_0 \big\downarrow & & \phi_1 \big\downarrow \\ J(f,x_0)\times G & \longrightarrow & J(f,x_1)\times G \end{array}$$

□

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