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PROPERTIES OF THE REIDEMEISTER NUMBERS ON TRANSFORMATION GROUPS

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Abstract. Let (X, G) be a transformation group and $\sigma(X, x_0, G)$ the fundamental group of (X, G). In this paper, we prove that the Reidemeister number $R(f_G)$ for an endomorphism $f_G : (X, G) \to (X, G)$ is a homotopy invariant. In particular, when any self-map $f : X \to X$ is homotopic to the identity map, we give some calculation of the lower bound of $R(f_G)$. Finally, we discuss commutativity and product formula for the Reidemeister number $R(f_G)$.

1. Introduction

In [5], F. Rhodes represented the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G), a group G of homeomorphisms of a space X, as a generalization of the fundamental group $\pi_1(X, G)$ of a topological space X. On the other hand, Ahn and Chung [1] studied the Reidemeister number for an endomorphism of a transformation group (X, G) as an extension of the Reidemeister number $R(f)$ for any self-map $f : X \to X$.

One objective of this paper is to show that the Reidemeister number $R(f_G)$ for an endomorphism of (X, G) is a homotopy invariance, and that the cardinality of the center of $\sigma(X, x_0, G)$ is an lower bound for the Reidemeister number $R(f_G)$ which any self-map $f : X \to X$ is homotopic to the identity map. In the second place, we prove the properties of the Reidemeister number $R(f_G)$ as follows : commutativity and product formula.

In this paper, we always assume that the spaces X and Y are compact connected polyhedra. The reader may refer to [5] for more de-

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tails on the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G).

2. Properties of the Reidemeister number

Let $f : X \to X$ be a self-map. In [5], if λ is a path from $f(x_0)$ to x_0, then λ induces an isomorphism

$$\lambda_* : \sigma(X, f(x_0), G) \to \sigma(X, x_0, G)$$

defined by $\lambda_*[\alpha; g] = [\lambda\rho + \alpha + g\lambda; g]$ for each $[\alpha; g] \in \sigma(X, f(x_0), G)$, where $\rho(t) = 1-t$. This isomorphism λ_* depends only on the homotopy class of λ.

In this section, we consider an endomorphism of (X, G). For the composition

$$\sigma(X, x_0, G) \xrightarrow{f_*} \sigma(X, f(x_0), G) \xrightarrow{\lambda_*} \sigma(X, x_0, G),$$

we denote $\lambda_* f_* = f_\sigma$. In [1], two elements $[\alpha; g_1]$ and $[\beta; g_2]$ of $\sigma(X, x_0, G)$ are said to be f_σ–equivalent, $[\alpha; g_1] \sim [\beta; g_2]$, if there exists $[\gamma; g] \in \sigma(X, x_0, G)$ such that

$$[\alpha; g_1] = [\gamma; g][\beta; g_2] f_\sigma([\gamma; g]^{-1}).$$

This is an equivalence relation on $\sigma(X, x_0, G)$. Let $\sigma(X, x_0, G)'(f_\sigma)$ be the set of equivalence classes of $\sigma(X, x_0, G)$ under f_σ–equivalence. The number of elements of the set $\sigma(X, x_0, G)'(f_\sigma)$ called the *algebraic Reidemeister number* of f_σ, denoted by $R_*(f_\sigma)$. With this definition, we may define the *Reidemeister number* of an endomorphism $f_G : (X, G) \to (X, G)$, $R(f_G)$, to be the algebraic Reidemeister number of f_σ, that is,

$$R(f_G) = R_*(f_\sigma).$$

LEMMA 1. *The definition of $R(f_G)$ is independent of the choice of the path λ from $f(x_0)$ to x_0 and the base-point $x_0 \in X$.*

Proof. (1) Independence of λ. Suppose that τ is another path from $f(x_0)$ to x_0. Then $\lambda^{-1}\tau$ is a loop at x_0. Since

$$\begin{aligned}(\lambda^{-1}\tau)_*([\alpha;g]) &= [\lambda^{-1}\tau\rho+\alpha+g\lambda^{-1}\tau;g]\\ &= [\lambda^{-1}\tau\rho;e][\alpha;g][\lambda^{-1}\tau;e],\end{aligned}$$

the loop $\lambda^{-1}\tau$ induces an inner automorphism

$$(\lambda^{-1}\tau)_* : \sigma(X,x_0,G)\to\sigma(X,x_0,G)$$

generated by the element $[\lambda^{-1}\tau;e]$.
Applying this automorphism to the left-hand side of λ_*f_*, we have

$$\begin{aligned}R_*(\lambda_*f_*) &= R_*(\tau_*\lambda_*^{-1}\lambda_*f_*)\\ &= R(\tau_*f_*).\end{aligned}$$

Hence we have independenceof the path λ.

(2) Independence of $x_0\in X$. For $x_1\in X$, let γ be a path from x_0 to x_1. Then $f\circ\gamma$ is a path from $f(x_0)$ to $f(x_1)$. Since γ and $f\circ\gamma$ induce isomorphisms γ_* and $(f\circ\gamma)_*$ respectively, we obtain the following commutative diagram :

$$\begin{array}{ccccc}\sigma(X,x_0,G) & \xrightarrow{f_*} & \sigma(X,f(x_0),G) & \xrightarrow{\lambda_*} & \sigma(X,x_0,G)\\ \gamma_*\downarrow & & (f\circ\gamma)_*\downarrow & & \gamma_*\downarrow\\ \sigma(X,x_1,G) & \xrightarrow{f'_*} & \sigma(X,f(x_1),G) & \xrightarrow{\lambda'_*} & \sigma(X,x_1,G)\end{array}$$

where λ' is a path from $f(x_1)$ to x_1. Since $\lambda_*=\gamma_*^{-1}\lambda'_*(f\circ\gamma)_*$ and $f_*=(f\circ\gamma)_*^{-1}f'_*\gamma_*$,

$$\begin{aligned}R_*(\lambda_*f_*) &= R_*(\gamma_*^{-1}\lambda'_*f'_*\gamma_*)\\ &= R_*(\lambda'_*f'_*).\end{aligned}$$

□

For a given homotopy $F:f\cong h:X\to X$ and a given path $c:I\to X$, define the diagonal path $\triangle(F,c):I\to X$ by $\triangle(F,c)(t)=F(c(t),t), 0\le t\le 1$. Let $\triangle^{-1}(F,c)$ denote the inverse of diagonal path $\triangle(F,c)$. Then the path $\triangle(F,c)$ preserves inverse in the following sence.

LEMMA 2. *[4]* $\triangle^{-1} (F, c) = \triangle (F^{-1}, c^{-1})$.

THEOREM 3. *(Homotopy invariance) Let* f_G *and* h_G *be endomorphisms of* (X, G). *If* $F : f \cong h : X \to X$ *is homotopy from* f *to* h, *then* $R(f_G) = R(h_G)$.

Proof. Let $x_0 \in X$. Then $\triangle (F, c)$ is a path from $f(x_0)$ to $h(x_0)$. Thus the path $\triangle (F, c)$ induces a homomorphism

$$\triangle (F, c)_* : \sigma(X, f(x_0), G) \to \sigma(X, h(x_0), G).$$

So we obtain the following induced commutative diagram

$$\begin{array}{ccc} \sigma(X, x_0, G) & \xrightarrow{f_*} & \sigma(X, f(x_0), G) \\ h_* \searrow & & \nearrow \triangle (F^{-1}, x_0)_* \\ & \sigma(X, h(x_0), G) & \end{array}$$

From Lemma 1 and Lemma 2, we have

$$\begin{aligned} R(f_G) &= R_*(\lambda_* f_*) \\ &= R_*(\lambda_* \triangle (F, x_0)_*^{-1} h_*) \\ &= R_*((\triangle (F^{-1}, x_0)\lambda)_* h_*) \\ &= R(h_G). \end{aligned}$$

□

THEOREM 4. *If a self-map* $f : X \to X$ *is homotopic to the identity map* id_X *of* X, *then*

$$R(f_G) = R(id_X) \geq |Z(\sigma(X, x_0, G))| \geq 1,$$

where $|Z(\sigma(X, x_0, G))|$ *is the number of elements of the center of* $\sigma(X, x_0, G)$.

Proof. Obiously, the first equality follows from Theorem 3. Since $Z(\sigma(X, x_0, G))$ contains at least the identity element $[x_0'; e]$, where x_0' is the constant map $x_0' : I \to X$, we have

$$|Z((X, x_0, G))| \geq 1.$$

Now we prove that $R(id_X) \geq |Z((X, x_0, G))|$. Consider

$$\sigma(X, x_0, G) \xrightarrow{id_*} \sigma(X, x_0, G) \xrightarrow{\lambda} \sigma(X, x_0, G).$$

For any element $[\alpha; g_1] \in \sigma(X, x_0, G)$, the id_X–equivalence class $\overline{[\alpha; g_1]}$ containing $[\alpha; g_1]$ is the set

$$\{[\gamma; g_2][\alpha; g_1]\lambda_*[\gamma; g_2]^{-1} | [\gamma; g_2] \in \sigma(X, x_0, G)\}.$$

Since λ is a loop at x_0,

$$\begin{aligned}\lambda_*([\gamma; g_2]^{-1}) &= \lambda_*([g_2^{-1}\gamma\rho; g_2^{-1}]) \\ &= [\lambda\rho; e][g_2^{-1}\gamma\rho; g_2^{-1}][\lambda; e] \\ &= [\lambda\rho; e][\gamma; g_2]^{-1}[\lambda; e].\end{aligned}$$

If $[\alpha; g_1] \in Z(\sigma(X, x_0, G))$, then the id_X–equivalence class consists of the single element $\lambda_*[\alpha; g_1]$, that is,

$$\begin{aligned}\overline{[\alpha; g_1]} &= \{[\lambda; e][\alpha; g_1][\lambda; e]\} \\ &= \{\lambda_*[\alpha; g_1]\}.\end{aligned}$$

Hence we have the desired result. □

THEOREM 5. *(Commutativity) Let f_G and h_G be endomorphisms of (X, G). Then*

$$R(f_G \circ h_G) = R(h_G \circ f_G).$$

Proof. From the following composition

$$\sigma(X, x_0, G) \xrightarrow{f_*} \sigma(X, f(x_0), G) \xrightarrow{h_*} \sigma(X, (h \circ f)(x_0), G),$$

we get $h_* \circ f_* = (h \circ f)_*$. Similarly, $f_* \circ h_* = (f \circ h)_*$. Let λ be a path from $(h \circ f)(x_0)$ to $(f \circ h)_*(x_0)$. Then λ induces an isomorphism

$$\lambda_* : \sigma(X, (h \circ f)(x_0), G) \to \sigma(X, (f \circ h)(x_0), G).$$

Thus we consider the following commutative diagram :

$$\begin{array}{ccc}
\sigma(X, x_0, G) & \xrightarrow{(h\circ f)_*} & \sigma(X, (h\circ f)(x_0), G) \\
{\scriptstyle (f\circ h)_*}\downarrow & & \downarrow{\scriptstyle \tau_*} \\
\sigma(X, (f\circ h)(x_0), G) & \xrightarrow{\gamma_*} & \sigma(X, x_0, G)
\end{array}$$

where τ is a path from $(h\circ f)(x_0)$ to x_0 and γ is a path from $(f\circ h)(x_0)$ to x_0.

Since $(f\circ h)_* = \lambda_*(h\circ f)_*$ and $\gamma_* = \tau_*\lambda_*^{-1}$, we have

$$\begin{aligned}
R(f_G \circ h_G) &= R((f\circ h)_G) \\
&= R_*(\gamma_*(f\circ h)_*) \\
&= R_*((\tau_*\lambda_*^{-1})(\lambda_*(h\circ f)_*)) \\
&= R_*(\tau_*(h\circ f)_*) \\
&= R(h_G \circ f_G).
\end{aligned}$$

Hence we complete the proof of this theorem. □

Let α_x be a path of order g with base–point x_0 in X, and α_y be a path of order h with base–point y_0 in Y. Then a path $\theta(\alpha_x, \alpha_y)$ of order (g, h) with base–point (x_0, y_0) in $X\times Y$ is defined by

$$\theta(\alpha_x, \alpha_y) = \begin{cases} (\alpha_x(2t), y_0), & 0 \le t \le \frac{1}{2}, \\ (gx_0, \alpha_y(2t-1)), & \frac{1}{2} \le t \le 1. \end{cases}$$

Note that we can see easily $(g, h)\theta(g\alpha_x, \alpha_y) = \theta(g\alpha_x, h\alpha_y)$ and $\theta(\alpha_x, \alpha_y)\rho = \theta(\alpha_x\rho, \alpha_y\rho)$, where $\rho(t) = 1 - t$. The homotopy class of $\theta(\alpha_x, \alpha_y)$ depends only on the homotopy classes of α_x and α_y. Hence θ induces an isomorphism

$$\theta_* : \sigma(X, x_0, G) \times \sigma(Y, y_0, H) \to \sigma(X\times Y, (x_0, y_0), G\times H)$$

$$\theta([\alpha_x; g], [\alpha_y; h]) = [\theta(\alpha_x, \alpha_y); (g, h)].$$

For an endomorphism $f'_H : (Y, H) \to (Y, H)$ and a homomorphism

$$f'_\sigma : \sigma(Y, y_0, H) \to \sigma(Y, y_0, H),$$

let $\sigma(Y, y_0, H)'(f'_\sigma)$ be the set of equivalence classes of $\sigma(Y, y_0, H)$ under f'_σ–equivalence.

Theorem 6. *(Product formula) Let f_G and f'_H be endomorphisms of (X, G) and (Y, H) respectively. Then*

$$R(f_G \times f'_H) = R(f_G) \cdot R(f'_H).$$

Proof. Note that if $[\alpha_x, g_1] \sim [\alpha'_x; g_2]$ and $[\alpha_y; h_1] \sim [\alpha'_y; h_2]$, then

$$[\theta(\alpha_x, \alpha_y); (g_1, h_1)] \sim [\theta(\alpha'_x, \alpha'_y); (g_2, h_2)].$$

The isomorphism θ_* induces an isomorphism

$$\begin{aligned} \overline{\theta_*} :& \sigma(X, x_0, G)'(f_\sigma) \times \sigma(Y, y_0, H)'(f'_\sigma) \to \\ & \sigma(X \times Y, (x_0, y_0), G \times H)'(f_\sigma \times f'_\sigma). \end{aligned}$$

Thus we obtain the following commutative diagram :

$$\begin{array}{ccc} \sigma(X, x_0, G) & \xrightarrow{\pi_1} & \sigma(X, x_0, G)'(f_\sigma) \times \sigma(Y, y_0, H)'(f'_\sigma) \\ {\scriptstyle \theta_*}\downarrow & & {\scriptstyle \overline{\theta_*}}\downarrow \\ \sigma(X \times Y, (x_0, y_0), G \times H) & \xrightarrow{\pi_2} & \sigma(X \times Y, (x_0, y_0), G \times H)'(f_\sigma \times f'_\sigma), \end{array}$$

where π_1 and π_2 are the natural projections. Hence

$$\begin{aligned} R(f_G \times f'_H) &= |\sigma(X \times Y, (x_0, y_0), G \times H)'(f_\sigma \times f'_\sigma)| \\ &= |\sigma(X, x_0, G)'(f_\sigma) \times \sigma(Y, y_0, H)'(f'_\sigma)| \\ &= |\sigma(X, x_0, G)'(f_\sigma)| \cdot |\sigma(Y, y_0, H)'(f'_\sigma)| \\ &= R(f_G) \cdot R(f'_H). \end{aligned}$$

□

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