Kangweon-Kyungki Math. Jour. **7** (1999), No. 2, pp. 171–181

THE EXISTENCE OF SOLUTIONS OF A NONLINEAR PARABOLIC EQUATION WITH NONLINEARITIES CROSSING EIGENVALUES

Q-Heung Choi and Zheng-Guo Jin

Abstract. We investigate multiplicity of solutions for a nonlinear perturbation of a parabolic operator under Dirichlet boundary condition, in a bounded domain.

1. Introduction

In this paper, we investigate the multiplicity of $u(x,t)$ for a nonlinear perturbation $f(u)$ of the parabolic operator $(L - D_t)$ under boundary condition on Ω and periodic condition on the variable t,

$$\begin{aligned} Lu - D_t u + f(u) &= h(x,t) && \text{in} \quad \Omega \times R, \\ u &= 0 && \text{on} \quad \partial\Omega, \\ u(x,t) &= u(x,t+T), \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$ and the nonlinear term $f(u)$ is piecewise linear one $bu^+ - au^-$ with $a < \lambda_{01} < b < \lambda_{02}$. Thus, we consider as a perturbation of the problem,

$$\begin{aligned} Lu - D_t u + bu^+ - au^- &= h(x,t) && \text{in } \Omega \times R, \\ u &= 0 && \text{on } \partial\Omega, \\ u(x,t) &= u(x,t+T). \end{aligned} \tag{1.2}$$

Here L is a second order elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact linear inverse, with eigenvalues $-\lambda_i$,

---

Received March 23, 1999.
1991 Mathematics Subject Classification: 35K20.
Key words and phrases: Multiplicity, Dirichlet boundary condition, eigenvalue.
This work supported in part by GARC and Korea Research Foundation 1998-015-D00044.

each repeated as often as multiplicity

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \longrightarrow +\infty.$$

Let H be the Hilbert space defined by

$$H = \{u \in L^2(\Omega \times [0,T]) | \quad u \text{ is } T\text{-periodic in } t \ \}$$

Then equation (1.2) is represented by

$$Lu - D_t u + bu^+ - au^- = h(x,t) \qquad \text{in} \qquad H. \tag{1.3}$$

In [Mc], the author showed by degree theory that equation (1.3) with the forcing term h is supposed to be a multiple of the first positive eigenfunction has at least two solutions if n is even, and at least three solutions if n is odd.

In this paper, we suppose that $a < \lambda_{01} < b < \lambda_{02}$ and the source term h is generated by φ_{01} and φ_{02}. Our goal is to investigate a relation between multiplicity of solution and source terms in equation (1.3) when h belongs to the two-dimensional subspace of space H spanned by φ_{01} and φ_{02}.

Let V be the two dimensional subspace of H spanned by φ_{01} and φ_{02}. Let P be the orthogonal projection H onto V. Let $\Phi : V \to V$ be a map defined by

$$\Phi(v) = Lv - D_t v + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), \quad v \in V.$$

In section 2, we suppose that the nonlinearity $-(bu^+ - au^-)$ crosses the eigenvalue λ_{01}. And we use the variational reduction method to reduce the problem from an infinite dimensional one to a finite dimensional one. In section 3, we investigate the properties of the map Φ and we reveal a relation between multiplicity of solution and source terms in equation (1.3) when h belongs to the two-dimensional space V.

2. A variational reduction

We consider the parabolic equation under the Dirichlet boundary condition and periodic condition on the variable t

$$\begin{aligned} Lu - D_t u + f(u) &= h(x,t) && \text{in } \Omega \times R, \\ u &= 0 && \text{on } \partial\Omega, \\ u(x,t) &= u(x,t+T), \end{aligned} \tag{2.1}$$

Here, the nonlinear term $f(u)$ is piecewise linear $bu^+ - au^-$ with $a < \lambda_{01} < b < \lambda_{02}$. Thus, we consider as a perturbation of the problem

$$\begin{aligned} Lu - D_t u + bu^+ - au^- &= h(x,t) && \text{in } \Omega \times R, \\ u &= 0 && \text{on } \partial\Omega, \\ u(x,t) &= u(x,t+T), \end{aligned} \tag{2.2}$$

where L is a second order elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact linear inverse, with eigenvalues $-\lambda_i$, each repeated according to its multiplicity,

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \longrightarrow +\infty.$$

We denote φ_n to be the eigenfunctions corresponding to eigenvalues λ_n and $\varphi_1(x) > 0$ in Ω. Let H be the Hilbert space defined by

$$H = \{u \in L^2(\Omega) \times [0,T] \;\mid\; u \text{ is } T\text{-periodic in } t\}.$$

Then the set $\{\varphi_{mn} = \frac{1}{\sqrt{2\pi}}\varphi_n(x)e^{imt}, n \geq 1, m = 0, \pm 1, \pm 2, \cdots\}$ is orthogonal in H and $\varphi_{01} > 0$

In this section, we suppose that $a < \lambda_{01} < b < \lambda_{02}$. Under this assumption, we are concerned with the multiplicity of solution of (2.2) only when h is generated by the eigenfunctions φ_{01} and φ_{02}. That is we study the equation

$$Lu - D_t u + bu^+ - au^- = h \qquad \text{in } H, \tag{2.3}$$

where $h = s_1\varphi_{01} + s_2\varphi_{02} (s_1, s_2 \in R)$.

Theorem 2.1. *If $s_1 < 0$, then (2.3) has no solution.*

Proof. We rewrite (2.3) as

$$(L - D_t + \lambda_{01})u + (b - \lambda_{01})u^+ - (a - \lambda_{01})u^- = s_1\varphi_{01} + s_2\varphi_{02}.$$

Multiply across by φ_{01} and integrate over H. Since $(L - D_t + \lambda_{01})\varphi_{01} = 0$ and $((L - D_t + \lambda_{01})u, \varphi_{01}) = 0$, thus we have

$$\int_\Omega \{(b-\lambda_{01})u^+ - (a-\lambda_{01})u^-\}\varphi_{01} = (s_1\varphi_{01} + s_2\varphi_{02}, \varphi_{01}) = s_1 \int_\Omega \varphi_{01}^2 = s_1.$$

However, we know that $(b - \lambda_{01})u^+ - (a - \lambda_{01})u^- \geq 0$ for all real valued function u. Also $\varphi_{01} > 0$ in H. Therefore

$$\int_\Omega \{(b - \lambda_{01})u^+ - (a - \lambda_{01})u^-\}\varphi_{01} \geq 0.$$

Hence, there is no solution of (2.3) if $s_1 < 0$. □

To study equation (2.3), we use the contraction mapping theorem to reduce the problem from an infinite dimensional one to a finite dimensional one.

Let V be two-dimensional subspace of H spanned by $\{\varphi_{01}, \varphi_{02}\}$ and W be the orthogonal complement of V in H. Let P be the orthogonal projection of H onto V. Then every $u \in H$ can be written as $u = v + w$, where $v = Pu$ and $w = (I - P)u$. Hence, equation (2.3) is equivalent to a system

$$Lw - D_t w + (I - P)(b(v + w)^+ - a(v + w)^-) = 0, \tag{2.4}$$

$$Lv - D_t v + P(b(v + w)^+ - a(v + w)^-) = s_1\varphi_{01} + s_2\varphi_{02}. \tag{2.5}$$

We look on this as a system of two equations in the two unknowns v and w.

LEMMA 2.2. *For a fixed $v \in V$, equation (2.4) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to the L^2-norm) in v.*

Proof. We use the contraction mapping theorem. Let $\delta = \frac{1}{2}(\lambda_{01}+\lambda_{02})$. Rewrite (2.4) as

$$(D_t - L - \delta)w = (I - P)(b(v + w)^+ - a(v + w)^- - \delta(v + w))$$

or equivalently

$$w = (D_t - L - \delta)^{-1}(I - P)g_v(w), \tag{2.6}$$

where $g_v(w) = b(v + w)^+ - a(v + w)^- - \delta(v + w)$. Since

$$\begin{aligned}|g_v(w_1) - g_v(w_2)| &= |b(v + w_1)^+ - a(v + w_1)^- - \delta(v + w_1) - b(v + w_2)^+ \\ &\quad - a(v + w_2)^- - \delta(v + w_2)| \\ &\le \max\{|b - \delta|, |\delta - a|\}|w_1 - w_2|,\end{aligned}$$

we have

$$\|g_v(w_1) - g_v(w_2)\| \le \max\{|b - \delta|, |\delta - a|\} \cdot \|w_1 - w_2\|.$$

Here $\|\cdot\|$ denotes the L^2-norm in H. The operator $(D_t - L - \delta)^{-1}(I - P)$ is a compact linear map from $(I - P)H$ into itself. If σ is the spectrum of $D_t - L$, then recall that $\sigma = \{\lambda_n \pm i2m, n \ge 1, m \ge 0\}$.The eigenvalues of the operator $T = (D_t - L - I\delta)^{-1}(I - P)$ in W are $(\lambda_n \pm i2m - \delta)^{-1}$and the operator norm of T, $\|T\| = |\lambda_{mn} - \delta|^{-1}$ for $n \ge 3$. For this to occur, we must require that there exists a circle C of radius r and center δ

such that C contains the points on the real line $a, b, \lambda_{01}, \lambda_{02}$ and does not contain any other points of the spectrum of $D_t - L$. Therefore its L^2-norm is

$$\|(D_t - L - \delta I)^{-1}(I - P)\| = \max(\text{dist}\{\delta, \lambda_n \pm i2m\} |\ m \geq 0, n \geq 3, n, m \in N)^{-1} = \frac{1}{\lambda_{03} - \delta}.$$

Since $\max\{|b - \delta|, |\delta - a|\} < \lambda_{03} - \delta$, it follows that for fixed $v \in V$, the right hand side of (2.6) defines a Lipschitz mapping W into itself with Lipschitz constant less than 1. Hence by the contraction mapping principle, for given $v \in V$, there is a unique $w \in W$ which satisfies (2.4). Also, it follows, by the standard argument principle that $\theta(v)$ is Lipschitz continuous in terms of v. □

By Lemma 2.2, the study of the multiplicity of solutions of (2.3) is reduced to the study of the multiplicity of solutions of an equivalent problem

$$Lv - D_t v + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) = s_1\varphi_{01} + s_2\varphi_{02} \qquad (2.7)$$

defined on the two dimensional subspace V spanned by $\{\varphi_{01}, \varphi_{02}\}$.

While one feels intuitively that (2.7) ought to be easier to solve than (2.3), there is the disadvantage of an implicitly defined term $\theta(v)$ in the equation. However, in our case, it turns out that we know $\theta(v)$ for some special c's.

Corollary 1. *If $v \geq 0$ or $v \leq 0$, then $\theta(v) \equiv 0$.*

Proof. Now , take $v \geq 0$ and $\theta(v) = 0$ since $v \in V, (I - P)v = 0$. Then equation (2.4) is reduced to

$$(L - D_t) \cdot 0 + (I - P)(bv^+ - av^-) = 0$$

because $v^+ = v, v^- = 0$ and $(I - P)v = 0$. By Lemma 2.2 , $\theta(v) \equiv 0$. □

Since $V = span\{\varphi_{01}, \varphi_{02}\}$ and φ_{01} is a positive eigenfunction, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\varphi_{01} + c_2\varphi_{02} |\quad c_1 \geq 0, |c_2| \leq \varepsilon_0 c_1\}$$

for some $\varepsilon_0 > 0$, so that $v \geq 0$ for all $v \in C_1$, and a cone C_3 defined by

$$C_3 = \{v = c_1\varphi_{01} + c_2\varphi_{02} |\quad c_1 \leq 0, |c_2| \leq \varepsilon_0 |c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$.Thus, we do not know $\theta(v)$ for all $v \in PH$, but we know $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$. And C_2 and C_4 one defined as follows

$$\begin{aligned} C_2 &= \{v = c_1\varphi_{01} + c_2\varphi_{02} | c_2 \geq 0, c_2 \geq \varepsilon_0 |c_1|\}, \\ C_4 &= \{v = c_1\varphi_{01} + c_2\varphi_{02} | c_2 \leq 0, |c_2| \geq \varepsilon_0 |c_1|\}. \end{aligned}$$

Then the union of C_1, C_3 and C_2, C_4 is the space V. Now we define a map $\Phi : V \longrightarrow V$ given by

$$\Phi(v) = Lv - D_t v + P(b(v + \theta(v))^+ - a(v + \theta(v))^-), v \in V.$$

Then Φ is continuous on V, since θ is continuous on V and we have the following Lemma.

Lemma 2.3. *For $v \in V$ and $c \geq 0, \Phi(cv) = c\Phi(v)$.*

Proof. Let $c \geq 0$. If v satisfies

$$L\theta(v) - D_t\theta(v) + (I - P)(b(v + \theta(v))^+ - a(v + \theta(v))^-) = 0 ,$$

then

$$L(c\theta(v) - D_t(c\theta(v)) + (I - P)(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-) = 0$$

and hence $\theta(cv) = c\theta(v)$. Therefore we have

$$\begin{aligned} \Phi(cv) &= L(cv) - D_t(cv) + P(b(cv + \theta(cv))^+ - a(cv + \theta(cv))^-) \\ &= L(cv) - D_t(cv) + P(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-) \\ &= cL(v) - cD_t v + cP(b(v + \theta(v))^+ - a(v + \theta(v))^-) \\ &= c\Phi(v). \end{aligned}$$

□

3. Multiplicity of solutions and source terms

Now we want to investigate the image of the cone C_1, C_3 under Φ. First we consider the image of C_1 under Φ. If $v = c_1\varphi_{01} + c_2\varphi_{02}$, then we have

$$\begin{aligned} \Phi(v) &= Lv - D_t v + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) \\ &= -c_1\lambda_{01}\varphi_{01} - c_2\lambda_{02}\varphi_{02} + b(c_1\varphi_{01} + c_2\varphi_{02}) \\ &= c_1(b - \lambda_{01})\varphi_{01} + c_2(b - \lambda_{02})\varphi_{02}. \end{aligned}$$

Thus the image of the rays $c_1\varphi_{01} \pm \varepsilon_0 c_2\varphi_{02} (c_1 \geq 0)$ can be explicitly calculated and they are

$$c_1(b - \lambda_{01})\varphi_{01} \pm \varepsilon_0 c_1(b - \lambda_{02})\varphi_{02} \quad (c_1 \geq 0).$$

Therefore if $a < \lambda_{01} < b < \lambda_{02}$,then Φ maps C_1 onto the cone

$$R_1 = \left\{ d_1\varphi_{01} + d_2\varphi_{02} | \quad d_1 \geq 0, |d_2| \leq \varepsilon_0 \left(\frac{\lambda_{02} - b}{b - \lambda_{01}} \right) d_1 \right\}.$$

Second, we consider the image of C_3. If $v = -c_1\varphi_{01} + c_2\varphi_{02} \leq 0 \quad (c_1 \geq 0, |c_2| \leq \varepsilon_0 c_1)$, then we have

$$\begin{aligned}
\Phi(v) &= Lv - D_t v + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) \\
&= Lv - D_t v + P(av) \\
&= c_1\lambda_{01}\varphi_{01} - c_2\lambda_{02}\varphi_{02} - ac_1\varphi_{01} + ac_2\varphi_{02} \\
&= c_1(\lambda_{01} - a)\varphi_{01} + c_2(a - \lambda_{02})\varphi_{02}.
\end{aligned}$$

Thus the image of the rays $-c_1\varphi_{01} \pm \varepsilon_0 c_1\varphi_{02}$ can be explicitly calculated and they are

$$c_1(\lambda_{01} - a)\varphi_{01} \pm \varepsilon_0 c_1(a - \lambda_{02})\varphi_{02} \ \ (c_1 \geq 0).$$

Therefore Φ maps C_3 onto the cone

$$R_3 = \left\{ d_1\varphi_{01} + d_2\varphi_{02} | \quad d_1 \geq 0, |d_2| \leq \varepsilon_0 \left(\frac{\lambda_{02} - a}{\lambda_{01} - a} \right) d_1 \right\}.$$

Here we consider the case $R_1 \subset R_3$. The relation $R_1 \subset R_3$ holds if and only if the nonlinearity $-(bu^+ - au^-)$ satisfies $b > \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$.

LEMMA 3.1. *For every $v = c_1\varphi_{01} + c_2\varphi_{02} \in V$, there exists a constant $d > 0$ such that $(\Phi(v), \varphi_{01}) \geq d_1|c_2|$.*

Proof. Let $g(u) = bu^+ - au^-$ and let $v = c_1\varphi_{01} + c_2\varphi_{02}$,

$$\begin{aligned}
\Phi(v) &= L(c_1\varphi_{01} + c_2\varphi_{02}) - D_t(c_1\varphi_{01} + c_2\varphi_{02}) \\
&\quad + P(g(c_1\varphi_{01} + c_2\varphi_{02} + \theta(c_1, c_2))) \\
&= L(c_1\varphi_{01} + c_2\varphi_{02}) + P(g(c_1\varphi_{01} + c_2\varphi_{02} + \theta(c_1, c_2))).
\end{aligned}$$

So if $u = c_1\varphi_{01} + c_2\varphi_{02} + \theta(c_1, c_2)$, then

$$\begin{aligned}
(\Phi((c_1\varphi_{01} + c_2\varphi_{02}), \varphi_{01}) &= ((L + \lambda_{01})(c_1\varphi_{01} + c_2\varphi_{02}), \varphi_{01}) \\
&\quad + ((g(u) - \lambda_{01}u), \varphi_{01}).
\end{aligned}$$

The first term is zero because $(L + \lambda_{01})\varphi_{01} = 0$ and L is a self-adjoint operator. The second term satisfies

$$\begin{aligned} g(u) - \lambda_{01}u &= bu^+ - au^- - \lambda_{01}(u^+ - u^-) \\ &= (b - \lambda_{01})u^+ + (\lambda_{01} - a)u^- \\ &\geq \gamma|u|, \end{aligned}$$

where $\gamma = \min\{b - \lambda_{01}, \lambda_{01} - a\} > 0$. Therefore we have

$$(\Phi(v), \varphi_{01}) \geq \gamma \int |u|\varphi_{01}.$$

Now there exists $d > 0$ so that $\gamma\varphi_{01} \geq d|\varphi_{02}|$ and therefore

$$\gamma \int |u|\varphi_{01} \geq d \int |u| \cdot |\varphi_{02}| \geq d|\int u\varphi_{02}| = d|c_2|,$$

which concludes the proof of Lemma. □

The Lemma 3.1 tells us that the image of Φ is contained in the right half-plane. That is, $\Phi(C_2)$ and $\Phi(C_4)$ are the cone in the right half-plane.

We consider the restriction $\Phi|_{C_i}(1 \leq i \leq 4)$ of Φ to the cone C_i. Let $\Phi_i = \Phi|_{c_i}(0 \leq i \leq 4)$ i.e.,

$$\Phi_i : C_i \longrightarrow V.$$

First, we consider Φ_1. It maps C_1 onto R_1. Let l_1 be the segment defined by

$$l_1 = \left\{\varphi_{01} + d_2\varphi_{02}| \quad |d_2| \leq \varepsilon_0 \left(\frac{\lambda_{02} - b}{b - \lambda_{01}}\right)\right\}.$$

Then the inverse image $\Phi^{-1}(l_1)$ is the segment

$$L_1 = \Phi_1^{-1}(l_1) = \left\{\frac{1}{b - \lambda_{01}}(\varphi_{01} + c_2\varphi_{02})| \quad |c_2| \leq \varepsilon_0\right\}.$$

By Lemma 2.3, $\Phi_1 : C_1 \longrightarrow R_1$ is bijective.

Next we consider Φ_3. It maps C_3 onto R_3. Let l_3 be the segment defined by

$$l_3 = \left\{\varphi_{01} + d_2\varphi_{02}| \quad |d_2| \leq \varepsilon_0 \left(\frac{\lambda_{02} - a}{a - \lambda_{01}}\right)\right\}.$$

Then the inverse image $\Phi_3^{-1}(l_3)$ is the segment

$$L_3 = \Phi_3^{-1}(l_3) = \left\{\frac{1}{a - \lambda_{01}}(\varphi_{01} + c_2\varphi_{02})| \quad |c_2| \leq \varepsilon_0\right\}.$$

By Lemma 2.3, $\Phi_3 : C_3 \longrightarrow R_3$ is bijective.

From now on, our goal is to find the images of the cones C_2 and C_4 under Φ, where

$$\begin{aligned} C_2 &= \{v = c_1\varphi_{01} + c_2\varphi_{02} | \quad c_2 \geq 0, \varepsilon_0|c_1| \leq c_2\}, \\ C_4 &= \{v = c_1\varphi_{01} + c_2\varphi_{02} | \quad c_2 \leq 0, \varepsilon_0|c_1| \leq |c_2|\}. \end{aligned}$$

By Theorem 2.1 and Lemma 2.2, the image of C_2 under Φ is a cone containing

$$R_2 = \left\{ d_1\varphi_{01} + d_2\varphi_{02} \mid d_1 \geq 0, \varepsilon_0 \left(\tfrac{\lambda_{02}-b}{b-\lambda_{01}}\right) d_1 \leq d_2 \leq \varepsilon_0 \left(\tfrac{\lambda_{02}-a}{\lambda_{01}-a}\right) d_1 \right\}$$

and the image of C_4 under Φ is a cone containing

$$R_4 = \left\{ d_1\varphi_{01} + d_2\varphi_{02} \mid d_1 \geq 0, -\varepsilon_0 \left(\tfrac{\lambda_{02}-a}{\lambda_{01}-a}\right) d_1 \leq d_2 \leq -\varepsilon_0 \left(\tfrac{\lambda_{02}-b}{b-\lambda_{01}}\right) d_1 \right\}.$$

We consider the restrictions Φ_2 and Φ_4, and define the segments l_2, l_4 as follows:

$$l_2 = \left\{ \varphi_{01} + d_2\varphi_{02} | \quad \varepsilon_0 \left(\frac{\lambda_{02}-b}{b-\lambda_{01}}\right) \leq d_2 \leq \varepsilon_0 \left(\frac{\lambda_{02}-a}{\lambda_{01}-a}\right) \right\},$$

$$l_4 = \left\{ \varphi_{01} + d_2\varphi_{02} | \quad \varepsilon_0 \left(\frac{a-\lambda_{02}}{\lambda_{01}-a}\right) \leq d_2 \leq \varepsilon_0 \left(\frac{b-\lambda_{02}}{b-\lambda_{01}}\right) \right\}.$$

We investigate the inverse image $\Phi_2^{-1}(l_2)$ and $\Phi_4^{-1}(l_4)$. Hence, we want to prove that Φ_2 and Φ_4 are surjective.

LEMMA 3.2. *Let $\gamma_i (i = 2, 4)$ be any simple path in R_i with end points on ∂R_i, where each ray (starting from the origin) in R_i intersects only one point of γ_i. Then the inverse image $\Phi_i^{-1}(\gamma_i)$ of γ_i is also a simple path in C_i with end points on ∂C_i, where any ray (starting from the origin) in C_i intersects only one point of this path.*

Proof. We note that $\Phi_i^{-1}(\gamma_i)$ is closed since Φ is continuous and γ_i is closed in V.

Suppose that there is a ray (starting from the origin) in C_i which intersects two points of $\Phi_i^{-1}(\gamma_i)$, say p and $\alpha p (\alpha > 1)$. Then, by Lemma 2.3 $\Phi_i(\alpha p) = \alpha\Phi_i(p)$ which implies that $\Phi_i(p) \in \gamma_i$ and $\Phi_i(\alpha p) \in \gamma_i$. This contradicts the assumption that each ray (starting from the origin) in C_i intersects only one point of γ_i.

We regard a point p as a radius vector in the plane V . Then for a point p in V, we define the argument *arg p* of p by the angle from the positive $\varphi_{01} - axis$ to p.

We claim that $\Phi_i^{-1}(\gamma_i)$ meets all the rays (starting from the origin) in C_i . If not, $\Phi_i^{-1}(\gamma_i)$ is disconnected in C_i. Since $\Phi_i^{-1}(\gamma_i)$ is closed and meet at most one point of any ray in C_i , there are two points p_1 and p_2 in C_2 such that $\Phi_i^{-1}(\gamma_i)$ does not contain any point $p \in C_i$ with

$$arg\ p_1 < arg\ p < arg\ p_2.$$

On the other hand, if we set l be the segment with end points p_1 and p_2. then $\Phi_i(l)$ is a path in R_i, where $\Phi_i(p_1)$ and $\Phi_i(p_2)$ belong to γ_i. Choose a point q in $\Phi_i(l)$ such that $arg\ q$ is between $arg\ \Phi_i(p_1)$ and $arg\ \Phi_i(p_2)$. Then there exist a point q' of γ_i such that $q' = \beta q$ for some $\beta > 0$. Hence $\Phi_i^{-1}(q)$ and $\Phi_i^{-1}(q')$ are on the same ray (starting from the origin) in C_i and

$$arg\ p_1 < arg\ \Phi_i^{-1}(q') < arg\ p_2\ ,$$

which is a contraction . This completes the proof. □

Lemma 3.2 implies that $\Phi_i (i = 2, 4)$ is surjective. Hence we have the following theorem.

THEOREM 3.3. *For $1 \leq i \leq 4$, the restriction Φ_i maps C_i onto R_i. Therefore, Φ maps V onto R_3. In particular, Φ_1 and Φ_3 are bijective.*

The above theorem also implies the following result.

THEOREM 3.4. *Suppose $a < \lambda_{01} < b < \lambda_{02}$ and $b > \frac{2\lambda_{01}\lambda_{02} - a(\lambda_{01}+\lambda_{02})}{\lambda_{01}+\lambda_{02}-2a}$. Let $h = s_1\varphi_{01} + s_2\varphi_{02}$. Then we have*

(1) If $h \in \bar{R}_1$, then (2.3) has exactly two solutions, one of which is positive and the other is negative.

(2) If h belongs to interior of R_2 or interior of R_4, then (2.3) has a negative solution and at least one sign changing solution.

(3) If h belongs to boundary of R_3, then (2.3) has a negative solution.

(4) If h does not belong to R_3, then (2.3) has no solution.

References

[1] A. Ambrosetti and G. Prodi, *A primer of nonlinear analysis*, vol.**34**. Cambridge, University Press, Cambridge Studies in Advanced Math. 1993.

[2] Robert F. Brown, *A topological introduction to nonlinear analysis,* Birkhaeuser,1993.

[3] Q.H. Choi and T.S. Jung, *An application of a variational reduction method to a nonlinear wave equation,* J. Differential Equations **117**(1995), 390-410.

[4] Q.H. Choi, S. Chun and T. Jung, *The multiplicity of solutions and geometry of a nonlinear ellliptic equation*, Studia Math., **120** (1996), 259-270.

[5] Q.H. Choi and H. Nam, *A nonlinear beam equation with nonlinearity crossing an eigenvalue*, J. Korean Math. Soc.**34**(1997),609-622.

[6] E. Kreyszig, *Introductory Functional Analysis with Applications* , John Wiley and Sons Inc, 1978.

[7] P.J.McKenna, *Topological Methods for Asymmetric Boundary Value Problems,* Lecture Notes Ser. 11, Res, Inst, Math, Global Analysis Res. Center, Seoul National University, 1993.

[8] P.J. McKenna, R. Redlinger and W. Walter,*Multiplicity results for asymptotically homogeneous semilinear boundary value problems*, Ann.Mat. Pura Appl.(4)**143** (1988),347-257.

[9] J. Schröder, *Operator Inequalities,* Academic Press, New York, 1980.

Department of Mathematics
Inha University
Incheon 402-751, Korea