

CONFORMAL CHANGE OF THE TENSOR $U^\nu{}_{\lambda\mu}$ IN 5-DIMENSIONAL g -UFT

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ABSTRACT. We investigate change of the tensor $U^\nu{}_{\lambda\mu}$ induced by the conformal change in 5-dimensional g -unified field theory. These topics will be studied for the second class in 5-dimensional case.

1. Introduction

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ([8],1957). CHUNG([6],1968) also investigated the same topic in 4-dimensional $*g$ -unified field theory.

The Einstein's connection induced by the conformal change for all classes in 3-dimensional case, for the second and third classes in 5-dimensional case, and for the first class in 5-dimensional $*g$ -UFT, and for the second class in 6-dimensional g -UFT were investigated by CHO ([1],1992, [2],1994, [3],1995, [4],1998).

In the present paper, we investigate change of the tensor $U^\nu{}_{\lambda\mu}$ induced by the conformal change in 5-dimensional g -unified field theory. These topics will be studied for the second class in 5-dimensional case.

2. Preliminaries

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may be referred to CHUNG([5],1988; [3],1988), CHO([1],1992; [2],1994; [3],1995).

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2.1. n -dimensional g -unified field theory

The n -dimensional g -unified field theory (n - g -UFT hereafter) was originally suggested by HLAVATÝ([8],1957) and systematically introduced by CHUNG([7],1963).

Let X_n ¹ be an n -dimensional generalized Riemannian manifold, referred to a real coordinate system x^ν obeying coordinate transformations $x^\nu \rightarrow x^{\nu'}$, for which

$$(2.1) \quad \text{Det} \left(\left(\frac{\partial x}{\partial x'} \right) \right) \neq 0.$$

In the usual Einstein's n -dimensional unified field theory, the manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$ ² :

$$(2.2) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$$

where

$$(2.3) \quad \text{Det}((g_{\lambda\mu})) \neq 0 \quad \text{Det}((h_{\lambda\mu})) \neq 0.$$

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$(2.4) \quad h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu.$$

In our n - g -UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in X_n in the usual manner.

The manifold X_n is connected by a general real connection $\Gamma_{\omega\mu}^\nu$ with the following transformation rule :

$$(2.5) \quad \Gamma_{\omega'\mu'}^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha} \left(\frac{\partial x^\beta}{\partial x^{\omega'}} \cdot \frac{\partial x^\gamma}{\partial x^{\mu'}} \Gamma_{\beta\gamma}^\alpha + \frac{\partial^2 x^\alpha}{\partial x^{\omega'} \partial x^{\mu'}} \right)$$

and satisfies the system of Einstein's equations

$$(2.6) \quad D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}$$

¹Throughout the present paper, we assumed that $n \geq 2$.

²Throughout this paper, Greek indices are used for holonomic components of tensors. In X_n all indices take the values $1, \dots, n$ and follow the summation convention.

where D_ω denotes the covariant derivative with respect to $\Gamma^\nu{}_{\lambda\mu}$ and

$$(2.7) \quad S_{\lambda\mu}{}^\nu = \Gamma^\nu{}_{[\lambda\mu]}$$

is the *torsion tensor* of $\Gamma^\nu{}_{\lambda\mu}$. The connection $\Gamma^\nu{}_{\lambda\mu}$ satisfying (2.6) is called the *Einstein's connection*.

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \dots$ are frequently used :

$$(2.8)a \quad \mathfrak{g} = \text{Det}((g_{\lambda\mu})) \neq 0, \quad \mathfrak{h} = \text{Det}((h_{\lambda\mu})) \neq 0, \\ \mathfrak{t} = \text{Det}((k_{\lambda\mu})),$$

$$(2.8)b \quad g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{t}}{\mathfrak{h}},$$

$$(2.8)c \quad K_p = k_{[\alpha_1}{}^{\alpha_1} \dots k_{\alpha_p]}{}^{\alpha_p}, \quad (p = 0, 1, 2, \dots)$$

$$(2.8)d \quad {}^{(0)}k_\lambda{}^\nu = \delta_\lambda^\nu, \quad {}^{(1)}k_\lambda{}^\nu = k_\lambda{}^\nu, \quad {}^{(p)}k_\lambda{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu,$$

$$(2.8)e \quad K_{\omega\mu\nu} = \nabla_\nu k_{\omega\mu} + \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu},$$

$$(2.8)f \quad \sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols $\{\Gamma^\nu{}_{\lambda\mu}\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

$$(2.9)a \quad K_0 = 1; K_n = k \text{ if } n \text{ is even}; \quad K_p = 0 \text{ if } p \text{ is odd},$$

$$(2.9)b \quad g = 1 + K_2 + \dots + K_{n-\sigma},$$

$$(2.9)c \quad {}^{(p)}k_{\lambda\mu} = (-1)^{p(p)}k_{\mu\lambda}, \quad {}^{(p)}k^{\lambda\nu} = (-1)^{p(p)}k^{\nu\lambda}.$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by T :

$$(2.10)a \quad \overset{pqr}{T} = \overset{pqr}{T}_{\omega\mu\nu} = T_{\alpha\beta\gamma} {}^{(p)}k_{\omega}{}^{\alpha(q)}k_{\mu}{}^{\beta(r)}k_{\nu}{}^{\gamma},$$

$$(2.10)b \quad T = T_{\omega\mu\nu} = \overset{000}{T},$$

$$(2.10)c \quad 2\overset{pqr}{T}_{\omega[\lambda\mu]} = \overset{pqr}{T}_{\omega\lambda\mu} - \overset{pqr}{T}_{\omega\mu\lambda},$$

$$(2.10)d \quad 2\overset{(pq)r}{T}_{\omega\lambda\mu} = \overset{pqr}{T}_{\omega\lambda\mu} + \overset{qpr}{T}_{\omega\lambda\mu}.$$

We then have

$$(2.11) \quad \overset{pqr}{T}_{\omega\lambda\mu} = -\overset{qpr}{T}_{\lambda\omega\mu}.$$

If the system (2.6) admits $\Gamma_{\lambda\mu}^{\nu}$, using the above abbreviations it was shown that the connection is of the form

$$(2.12) \quad \Gamma_{\omega\mu}^{\nu} = \{\omega_{\mu}^{\nu}\} + S_{\omega\mu}{}^{\nu} + U^{\nu}{}_{\omega\mu}$$

where

$$(2.13) \quad U_{\nu\omega\mu} = 2\overset{001}{S}_{\nu(\omega\mu)}.$$

The above two relations show that *our problem of determining $\Gamma_{\omega\mu}^{\nu}$ in terms of $g_{\lambda\mu}$ is reduced to that of studying the tensor $S_{\omega\mu}{}^{\nu}$* . On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^{\nu}$ satisfies

$$(2.14) \quad S = B - 3\overset{(110)}{S}$$

where

$$(2.15) \quad 2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}k_{\omega]}{}^{\alpha}k_{\nu}{}^{\beta}.$$

2.2. Some results for the second class in 5-g-UFT

In this section, we introduce some results of 5-g-UFT without proof, which are needed in our subsequent considerations.

They may be referred to CHO([1],1992).

DEFINITION 2.1. In 5-g-UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be the second class, if $K_2 \neq 0, K_4 = 0$.

THEOREM 2.2. (Main recurrence relations) For the second class in 5-UFT, the following recurrence relation hold

$$(2.16) \quad {}^{(p+3)}k_\lambda{}^\nu = -K_2 {}^{(p+1)}k_\lambda{}^\nu, \quad (p = 0, 1, 2, \dots).$$

THEOREM 2.3. (For the second class in 5-g-UFT). A necessary and sufficient condition for the existence and uniqueness of the solution of (2.5) is

$$(2.17) \quad 1 - (K_2)^2 \neq 0.$$

If the condition (2.17) is satisfied, the unique solution of (2.14) is given by

$$(2.18) \quad (1 - K_2^2)(S - B) = -2 \overset{(10)1}{B} + (K_2 - 1) \overset{110}{B} + 2 \overset{(20)2}{B} + 2 \overset{112}{B}.$$

3. Conformal change of the 5-dimensional tensor $U^\nu_{\lambda\mu}$ for the second class.

In this final chapter we investigate the change $U^\nu_{\lambda\mu} \rightarrow \bar{U}^\nu_{\lambda\mu}$ of the tensor induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that X_n and \bar{X}_n are conformal if and only if

$$(3.1) \quad \bar{g}_{\lambda\mu}(x) = e^\Omega g_{\lambda\mu}(x)$$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the tensor $U^\nu_{\lambda\mu}$. An explicit representation of the change of 5-dimensional tensor $U^\nu_{\lambda\mu}$ for the second class will be exhibited in this chapter.

AGREEMENT (3.1). Throughout this section, we agree that, if T is a function of $g_{\lambda\mu}$, then we denote \bar{T} the same function of $\bar{g}_{\lambda\mu}$. In particular, if T is a tensor, so is \bar{T} . Furthermore, the indices of T (\bar{T}) will be raised and/or lowered by means of $h^{\lambda\nu}(\bar{h}^{\lambda\nu})$ and/or $h_{\lambda\mu}(\bar{h}_{\lambda\mu})$.

The results in the following theorems are needed in our further considerations. They may be referred to CHO([1],1992, [2],1994, [3],1998).

THEOREM 3.2. *In n - g -UFT, the conformal change (3.1) induces the following changes :*

$$(3.2)a \quad \begin{aligned} {}^{(p)}\bar{k}_{\lambda\mu} &= e^{\Omega(p)} k_{\lambda\mu}, & {}^{(p)}\bar{k}_{\lambda}{}^{\nu} &= {}^{(p)}k_{\lambda}{}^{\nu}, \\ {}^{(p)}\bar{k}^{\lambda\nu} &= e^{-\Omega(p)} k^{\lambda\nu} \end{aligned}$$

$$(3.2)b \quad \bar{g} = g, \quad \bar{K}_p = K_p, \quad (p = 1, 2, \dots).$$

Now, we are ready to derive representations of the changes $U^{\nu}{}_{\lambda\mu} \rightarrow \bar{U}^{\nu}{}_{\lambda\mu}$ in 5- g -UFT for the second class induced by the conformal change (3.1).

THEOREM 3.3. *The change $S_{\lambda\mu}{}^{\nu} \rightarrow \bar{S}_{\lambda\mu}{}^{\nu}$ induced by conformal change (3.1) may be represented by*

$$(3.3) \quad \begin{aligned} \bar{S}_{\lambda\mu}{}^{\nu} &= S_{\lambda\mu}{}^{\nu} + \frac{1}{C} [(3 - K_2 + K_2^2)^{(2)} k^{\nu}{}_{[\lambda} k_{\mu]}{}^{\delta} \Omega_{\delta} \\ &+ 2K_2^2 k^{\nu}{}_{[\lambda} k_{\mu]}{}^{\delta} \Omega_{\delta} + (4K_2 + 2K_2^2) k^{\nu}{}_{[\lambda} {}^{(2)}k_{\mu]}{}^{\delta} \Omega_{\delta} \\ &+ (1 - K_2 + 4K_2^2) k_{\lambda\mu} {}^{(2)}k^{\nu\delta} \Omega_{\delta} + (-1 - K_2) k_{\lambda\mu} \Omega^{\nu} \\ &+ (1 + K_2) k^{\nu}{}_{[\lambda} \Omega_{\mu]} + (-1 - K_2^2) h^{\nu}{}_{[\lambda} k_{\mu]}{}^{\delta} \Omega_{\delta}] \end{aligned}$$

where $C = K_2^2 - 1$.

THEOREM 3.4. *The change $U^{\nu}{}_{\lambda\mu} \rightarrow \bar{U}^{\nu}{}_{\lambda\mu}$ induced by the conformal change (3.1) may be represented by*

$$(3.4) \quad \begin{aligned} \bar{U}^{\nu}{}_{\lambda\mu} &= U^{\nu}{}_{\lambda\mu} + \frac{1}{C} \{ C_1 k^{\nu}{}_{(\lambda} k_{\mu)}{}^{\delta} \Omega_{\delta} \\ &+ C_2 [{}^{(2)}k^{\nu}{}_{(\lambda} {}^{(2)}k_{\mu)}{}^{\delta} \Omega_{\delta} + {}^{(2)}k_{\lambda\mu} {}^{(2)}k^{\nu\delta} \Omega_{\delta}] \\ &+ C_3 [{}^{(2)}k^{\nu}{}_{(\lambda} \Omega_{\mu)} - {}^{(2)}k_{\lambda\mu} \Omega^{\nu}] \} \end{aligned}$$

where $C_1 = -6K_2^3 + 2K_2^2 - K_2 - 1$, $C_2 = 2K_2(K_2 + 2)$, $C_3 = 1 + K_2$.

Proof. In virtue of (2.13) and Agreement (3.1), we have

$$(3.5) \quad \bar{U}_{\nu\lambda\mu} = \overline{S^{\frac{100}{(\lambda\mu)\nu}}}$$

The relation (3.4) follows by substituting (3.3), (2.10), Definition (2.1), (2.16), (3.2) into (3.5). \square

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