A NOTE ON SIMPLE SINGULAR
GP-INJECTIVE MODULES

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Abstract. We investigate characterizations of rings whose simple
singular right $R$-modules are $GP$-injective. It is proved that if $R$ is
a semiprime ring whose simple singular right $R$-modules are $GP$-
injective, then the center $Z(R)$ of $R$ is a von Neumann regular ring.
We consider the condition $(*)$: $R$ satisfies $l(a) \subseteq r(a)$ for any $a \in R$.
Also it is shown that if $R$ satisfies $(*)$ and every simple singular right
$R$-module is $GP$-injective, then $R$ is a reduced weakly regular ring.

1. Introduction

Throughout this paper $R$ denotes an associative ring with identity,
and all modules are unitary right $R$-modules. For any nonempty subset
$X$ of a ring $R$, $r(X)$ and $l(X)$ denote the right annihilator of $X$ and
the left annihilator of $X$, respectively. A right $R$-module $M$ is called
generalized right principally injective (briefly right $GP$-injective) if,
for any $0 \neq a \in R$, there exists a positive integer $n$ such that $a^n \neq
0$ and any right $R$-homomorphism of $a^nR$ into $M$ extends to one of
$R$ into $M$. This concept was introduced by Yuechiming [11]. Von
Neumann regularity of rings whose simple right $R$-modules are injective
was studied by many authors in [4], [6], [9] and [10]. Recently, we
proved that a ring $R$ is strongly regular iff $R$ is a right quasi-duo ring
whose simple right $R$-modules are $GP$-injective [6]. Rings whose simple
singular right $R$-modules are injective were studied by many authors
in [1], [2] and [8]. Yuechiming proved that a ring $R$ is strongly regular
iff $R$ is a semiprime right quasi-duo ring whose simple singular right
$R$-modules are $p$-injective [10]. Recently, Ding and Chen proved that a
ring $R$ is strongly regular iff $R$ is a right duo ring whose simple singular
right $R$-modules are $GP$-injective [3]. We also proved that a ring $R$ is

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strongly regular iff $R$ is an abelian right quasi-duo ring whose simple singular right $R$-modules are $GP$-injective [5].

In this paper we consider rings whose simple singular right $R$-modules are $GP$-injective. We investigate characterizations of rings whose every simple singular right $R$-module is $GP$-injective. Actually we prove the following facts: 1. If $R$ is a semiprime ring whose simple singular right $R$-modules are $GP$-injective, then the center $Z(R)$ of $R$ is a von Neumann regular ring. 2. Assume that every simple singular right $R$-module is $GP$-injective. If $R$ satisfies $(\ast)$, then $R$ is a reduced weakly regular ring.

2. Rings whose simple singular modules are $GP$-injective

We begin with the following lemmas.

**Lemma 1.** If $R$ is a semiprime ring, then $r(a^n) = r(a)$ for any $a \in Z(R)$ and $n \in \mathbb{Z}^+$, where $Z(R)$ denotes the center of $R$.

**Proof.** It can be easily verified by using induction. \qed

The following lemma is well-known, so we omit its proof.

**Lemma 2.** For any $a \in Z(R)$, if $a = ara$ for some $r \in R$, then there exists $b \in Z(R)$ such that $a = aba$.

**Proposition 3.** If $R$ is a semiprime ring whose simple singular right $R$-modules are $GP$-injective, then the center $Z(R)$ of $R$ is a von Neumann regular ring.

**Proof.** First we will show that $aR + r(a) = R$ for any $a \in Z(R)$. If not, there exists a maximal right ideal $M$ of $R$ such that $aR + r(a) \subseteq M$. Since $a \in Z(R)$, $aR + r(a)$ is an essential right ideal and so $M$ must be an essential right ideal of $R$. Therefore $R/M$ is $GP$-injective. So there exists a positive integer $n$ such that any $R$-homomorphism of $a^nR$ into $R/M$ extends to one of $R$ into $R/M$. Let $f : a^nR \to R/M$ be defined by $f(a^n r) = r + M$. Since $R$ is semiprime, by Lemma 1 $f$ is a well-defined $R$-homomorphism. Now $R/M$ is $GP$-injective, so there exists $c \in R$ such that $1 + M = f(a^n) = ca^n + M$. Hence $1 - ca^n \in M$ and so $1 \in M$, which is a contradiction. Therefore $aR + r(a) = R$ for any
$a \in Z(R)$ and so we have $a = ara$ for some $r \in R$. Applying Lemma 2, $Z(R)$ is a von Neumann regular ring.

Recall that a ring $R$ is right (left) weakly regular if $I^2 = I$ for each right (left) ideal $I$ of $R$; equivalently, $a \in aRaR (a \in RaRa)$ for every $a \in R$. $R$ is weakly regular if it is both right and left weakly regular [7]. Rings whose simple right $R$-modules are $GP$-injective are always semiprime [6]. But in general rings whose simple singular right $R$-modules are injective (hence also $GP$-injective) need not be semiprime [8].

We consider the condition (*): $R$ satisfies $l(a) \subseteq r(a)$ for any $a \in R$.

**Lemma 4.** If $R$ satisfies (*), then $RaR + r(a)$ is an essential right ideal of $R$.

**Proof.** Given $a \in R$, assume that $[RaR + r(a)] \cap I = 0$ where $I$ is a right ideal of $R$. Then $Ia \subseteq I \cap RaR = 0$ and so $I \subseteq l(a) \subseteq r(a)$. Hence $I = 0$; whence $RaR + r(a)$ is an essential right ideal of $R$. □

**Lemma 5.** If $R$ satisfies (*) and every simple singular right $R$-module is $GP$-injective, then $R$ is a reduced.

**Proof.** Let $a^2 = 0$. Suppose that $a \neq 0$. By Lemma 4, $r(a)$ is an essential right ideal of $R$. Since $a \neq 0$, $r(a) \neq R$. Thus there exists a maximal essential right ideal $M$ of $R$ containing $r(a)$. Therefore $R/M$ is $GP$-injective. So any $R$-homomorphism of $aR$ into $R/M$ extends to one of $R$ into $R/M$. Let $f : aR \to R/M$ be defined by $f(ar) = r + M$. Clearly $f$ is a well-defined $R$-homomorphism. Thus $1 + M = f(a) = ca + M$. Hence $1 - ca \in M$ and so $1 \in M$, which is a contradiction. Hence $a = 0$, and so $R$ is reduced. □

**Theorem 6.** If $R$ satisfies (*) and every simple singular right $R$-module is $GP$-injective, then $R$ is a reduced weakly regular ring.

**Proof.** By Lemma 5, $R$ is a reduced ring. We will show that $RaR + r(a) = R$ for any $a \in R$. Suppose that there exists $b \in R$ such that $RbR + r(b) \neq R$. Then there exists a maximal right ideal $M$ of $R$ containing $RaR + r(b)$. By Lemma 4, $M$ must be essential in $R$. Therefore $R/M$ is $GP$-injective. So there exists a positive integer $n$ such that any $R$-homomorphism of $b^nR$ into $R/M$ extends to one of $R$ into $R/M$. Let
$f : b^n R \to R/M$ be defined by $f(b^n r) = r + M$. Since $R$ is a reduced ring, $f$ is a well-defined $R$-homomorphism. Now $R/M$ is $GP$-injective, so there exists $c \in R$ such that $1 + M = f(b^n) = cb^n + M$. Hence $1 - cb^n \in M$ and so $1 \in M$, which is a contradiction. Therefore $RaR + r(a) = R$ for any $a \in R$. Hence $R$ is a right weakly regular ring. Since $R$ is reduced, it is also can be easily verified that $R$ is a weakly regular ring. □

References