

ON THE NONHOLONOMIC FRAMES IN A GENERALIZED 2-DIMENSIONAL RIEMANNIAN MANIFOLD X_2

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ABSTRACT. We will derive some identities related to p-scalars, p-vectors, and non-holonomic frames in a generalized 2-dimensional Riemannian manifold X_2 .

1. Introduction

Let X_2 be a generalized 2-dimensional Riemannian manifold referred to a real coordinate system x^μ_λ and endowed with a real quadratic tensor $g_{\lambda\mu}$, which may be split into its symmetric part $h_{\lambda\mu}$ and skew symmetric part $k_{\lambda\mu}$:

$$(1.1) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$$

In this paper, we assume that

$$(1.2) \quad \mathfrak{g} = \text{Det}((g_{\lambda\mu})) \quad , \quad \mathfrak{h} = \text{Det}((h_{\lambda\mu})) < 0 \quad , \quad \mathfrak{k} = \text{Det}((k_{\lambda\mu}))$$

According to (1.2), there exists a unique symmetric tensor $h_{\lambda\mu}$ defined by

$$(1.3) \quad h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu$$

The tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising or lowering indices of tensor quantities in X_2 in the usual manner.

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The purpose of the present paper is, in the first, to study the properties of the p-scalars H and the corresponding eigenvectors P^ν in X_2 , defined by

$$(1.4) \quad (Hh_{\nu\lambda} - {}^{(p)}k_{\nu\lambda})P^\nu = 0, \quad (H : \text{a scalar})$$

Finally, we introduce a nonholonomic frame to X_2 and find expressions for the tensor $h_{\lambda\mu}$ and ${}^{(p)}k_{\nu\lambda}$ in terms of the p-vectors P^ν .

2. Preliminaries

This section is a brief collection of definitions, notations, and basic results which are needed in our subsequent considerations.

$$(2.1) \quad g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}}$$

$$(2.2) \quad \mathfrak{g} = \mathfrak{h} + \mathfrak{k}$$

$$(2.3) \quad {}^{(0)}k_\lambda^\nu = \delta_\lambda^\nu, \quad {}^{(p)}k_\lambda^\nu = {}^{(p-1)}k_\lambda^\alpha k_\alpha^\nu \quad (p = 1, 2, \dots)$$

$$(2.4) \quad \mathfrak{k} = \Omega^2 > 0, \quad k = \frac{\Omega^2}{\mathfrak{h}} < 0 \quad \text{where} \quad \Omega = k_{12}$$

$$(2.5a) \quad {}^{(p)}k_{\lambda\mu} = (-k)^{\frac{p}{2}} h_{\lambda\mu} \quad \text{if } p \text{ is even}$$

$$(2.5b) \quad {}^{(p)}k_{\lambda\mu} = (-k)^{\frac{p-1}{2}} k_{\lambda\mu} \quad \text{if } p \text{ is odd}$$

Furthermore, we use $\mathcal{E}^{\lambda\mu}(\mathfrak{e}_{\lambda\mu})$ as the contravariant (covariant) indicators of weight $1(-1)$.

The eigenvalue M and the corresponding eigenvector a^ν in X_2 defined by

$$(2.6) \quad (Mh_{\nu\lambda} - k_{\nu\lambda})a^\nu = 0 \quad (M : \text{a scalar})$$

are called basic scalars and basic vectors of X_2 , respectively.

There are exactly two linearly independent basic vectors a^ν satisfying (2.6), where the corresponding basic scalars M are given by

$$(2.7) \quad M = M_1 = -M_2 = \sqrt{-k}$$

3. P - Scalars

LEMMA 3.1. We have

$$(3.1) \quad \mathfrak{k} > 0 \quad , \quad k = \frac{\mathfrak{k}}{\mathfrak{h}} < 0$$

Proof. Since $k_{\lambda\mu}$ is skew symmetric, $\mathfrak{k} = (k_{12})^2 = \Omega^2 > 0$ $\square$

LEMMA 3.2. We have

$$(3.2) \quad \begin{aligned} A(H) &\stackrel{def}{=} Det((Hh_{\nu\lambda} - {}^{(p)}k_{\nu\lambda})) \\ &= \begin{cases} \mathfrak{h}(H - (-k)^{\frac{p}{2}})^2, & \text{if } p \text{ is even} \\ \mathfrak{h}(H^2 + k^p), & \text{if } p \text{ is odd.} \end{cases} \end{aligned}$$

Proof. (case 1) p is even:

$$\begin{aligned} 2A(H) &= 2Det((Hh_{\nu\lambda} - {}^{(p)}k_{\mu\beta})) \\ &= \mathcal{E}^{\omega\mu}\mathcal{E}^{\alpha\beta}(Hh_{\omega\alpha} - {}^{(p)}k_{\omega\alpha})(Hh_{\mu\beta} - {}^{(p)}k_{\mu\beta}) \\ &= 2H^2\mathfrak{h} - 2H\mathcal{E}^{\omega\mu}\mathcal{E}^{\alpha\beta}h_{\omega\alpha}{}^{(p)}k_{\mu\beta} + 2Det({}^{(p)}k_{\lambda\mu}) \\ &= 2H^2\mathfrak{h} - 2H(2(-k)^{\frac{p}{2}}\mathfrak{h}) + 2k^2\mathfrak{h} \\ &= 2\mathfrak{h}(H - (-k)^{\frac{p}{2}})^2 \end{aligned}$$

(case 2) p is odd:

$$\begin{aligned} 2A(H) &= 2Det((Hh_{\nu\lambda} - {}^{(p)}k_{\nu\lambda})) \\ &= \mathcal{E}^{\omega\mu}\mathcal{E}^{\alpha\beta}(Hh_{\omega\alpha} - {}^{(p)}k_{\omega\alpha})(Hh_{\mu\beta} - {}^{(p)}k_{\mu\beta}) \\ &= 2H^2\mathfrak{h} - 2H\mathcal{E}^{\omega\mu}\mathcal{E}^{\alpha\beta}h_{\omega\alpha}{}^{(p)}k_{\mu\beta} + 2Det({}^{(p)}k_{\lambda\mu}) \\ &= 2\mathfrak{h}(H^2 + k^p) \end{aligned} \quad \square$$

THEOREM 3.3. The eigenvalues H of (2.4) are given by

$$(3.3) \quad \begin{cases} H = (-k)^{\frac{p}{2}}, & p \text{ is even} \\ H = \pm(-k)^{\frac{p}{2}}, & p \text{ is odd} \end{cases}$$

Proof. The existence of the above case is clear from Lemma(3.1). A necessary and sufficient condition for the existence of a nontrivial solution P^ν of (1.4) is $A(H) = 0$. According to Lemma(3.2), the result follows directly. $\square$

REMARK. In case that p is even, $H = (-k)^{\frac{p}{2}}$ is a double root of $A(H) = 0$.

THEOREM 3.4. Every basic vector a_i^ν of X_2 is also p -vector of X_2 .

Proof. (case 1) p is even:

From (2.5a), we have

$${}^{(p)}k_{\nu\lambda}a_i^\nu = (-k)^{\frac{p}{2}}h_{\nu\lambda}a_i^\nu = M^p h_{\nu\lambda}a_i^\nu$$

where $M = \sqrt{-k}$.

(case 2) p is odd:

From (2.5b), we have

$${}^{(p)}k_{\nu\lambda}a_i^\nu = (-k)^{\frac{p-1}{2}}k_{\nu\lambda}a_i^\nu = (-k)^{\frac{p-1}{2}}Mh_{\nu\lambda}a_i^\nu = M^p h_{\nu\lambda}a_i^\nu$$

where $M = M_1 = M_2 = \sqrt{-k}$

In both cases, we have

$${}^{(p)}k_{\nu\lambda}a_i^\nu = Hh_{\nu\lambda}a_i^\nu$$

where $H = M^p$.

Therefore, a_i^ν is the p -vector P^ν with p -scalar H . $\square$

THEOREM 3.5. In case that p is even, every vector of X_2 is a p -vector of X_2 .

Proof. From (2.5a), the relation (1.4) may be written as

$$(3.4) \quad (H - (-k)^{\frac{p}{2}})h_{\nu\lambda}P^\nu = 0$$

Since $\mathfrak{h} \neq 0$, (3.4) holds for every vector P^ν with $H = (-k)^{\frac{p}{2}}$. $\square$

4. P-Vectors

In this section, we shall study and derive several properties of p-vectors given by (1.4).

THEOREM 4.1. *In case that p is odd, the p -vector P^ν is a null vector.*

Proof. Since the tensor ${}^{(p)}k_{\lambda\mu}$ is skew-symmetric and from (1.4), for $i = 1, 2$,

$$H h_{\nu\lambda} P_i^\nu P_i^\lambda = {}^{(p)}k_{\nu\lambda} P_i^\nu P_i^\lambda = -{}^{(p)}k_{\lambda\nu} P_i^\lambda P_i^\nu = -{}^{(p)}k_{\nu\lambda} P_i^\nu P_i^\lambda = 0.$$

In virtue of (3.3), $H \neq 0$, we have

$$h_{\nu\lambda} P_i^\nu P_i^\lambda = 0. \quad \square$$

THEOREM 4.2. *In case that p is odd, there are only two eigenvectors P_i^ν and P_i^λ which have the following properties.*

- (a) *They are defined up to arbitrary factor of proportionality.*
- (b) *They are null.*
- (c) *They are not perpendicular.*
- (d) *Their directions are real.*

Proof. We are only to prove (c).

$$H_2 P_2^\nu P_1^\nu = {}^{(p)}k_{\mu\lambda} P_1^\lambda P_2^\mu = -{}^{(p)}k_{\mu\lambda} P_1^\mu P_2^\lambda = -{}^{(p)}k_\mu^\nu P_1^\mu P_2^\mu = -H_1 P_1^\nu P_2^\nu$$

where $H_1 = (-k)^{\frac{p}{2}}$, $H_2 = -(-k)^{\frac{p}{2}}$. This is equivalent to

$$(H_1 + H_2) h_{\mu\lambda} P_1^\mu P_2^\lambda = 0.$$

Since $H_1 + H_2 = 0$, we have $h_{\mu\lambda} P_1^\mu P_2^\lambda$ is arbitrary, so that $h_{\mu\lambda} P_1^\mu P_2^\lambda \neq 0$. $\square$

THEOREM 4.3. *If p is even, P^ν is not necessarily null.*

Proof.

$$Hh_{\nu\lambda}P^\nu P^\lambda = {}^{(p)}k_{\nu\lambda}P^\nu P^\lambda = (-k)^{\frac{p}{2}}h_{\nu\lambda}P^\nu P^\lambda$$

so that

$$(H - (-k)^{\frac{p}{2}})h_{\nu\lambda}P^\nu P^\lambda = 0.$$

But

$$(H - (-k)^{\frac{p}{2}}) = 0.$$

Therefore, we have

$$h_{\nu\lambda}P^\nu P^\lambda \neq 0.$$

□

5. Nonholonomic Frames in X_2

For our further discussions we make the following agreement.

AGREEMENT. *The factor of proportionality mentioned in Theorem (4.2)a may be chosen in such a way that*

$$(5.1) \quad h_{\lambda\mu}P_1^\lambda P_2^\mu = 1.$$

Since P_1^λ and P_2^μ are linearly independent, there exists a unique reciprocal set of two linearly independent covariant vectors $\overset{1}{P}_\lambda$ and $\overset{2}{P}_\lambda$ such that

$$(5.2a) \quad P_i^\nu \overset{i}{P}_\lambda = \delta_\lambda^\nu.$$

These equations are equivalent to

$$(5.2b) \quad P_j^\lambda \overset{j}{P}_\lambda = \delta_j^i.$$

DEFINITION 5.1. With the vectors P^ν and $\overset{i}{P}_\lambda$, a nonholonomic frame of X_2 is defined in the following usual way : If $T_{\lambda\cdots}^{\nu\cdots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

$$(5.3a) \quad T_{j\cdots}^{i\cdots} \stackrel{def}{=} T_{\lambda\cdots}^{\nu\cdots} \overset{j}{P}_\nu \overset{i}{P}_\lambda \cdots$$

An easy inspection of (5.2)a and (5.1) shows

$$(5.3b) \quad T_{\lambda\cdots}^{\nu\cdots} \stackrel{def}{=} T_{j\cdots}^{i\cdots} P_i^\nu P_\lambda^j \cdots$$

THEOREM 5.1. *Let p be odd. Then the nonholonomic components*

$$(5.4) \quad h_{ij} = h_{\lambda\mu} \overset{\lambda}{P}_i^\lambda \overset{\mu}{P}_j^\mu \quad ; \quad h^{ij} = h^{\lambda\mu} \overset{i}{P}_\lambda^i \overset{j}{P}_\mu^j$$

are given by the matrix equation

$$(5.5) \quad ((h_{ij})) = ((h^{ij})) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Proof. From Theorem(4.2b) and (5.1), we have

$$(5.6) \quad h_{ij} h^{kj} = h_{\mu\lambda} \overset{\mu}{P}_i^\mu \overset{\lambda}{P}_j^\lambda h^{\nu\omega} \overset{k}{P}_\nu^k \overset{j}{P}_\omega^j = h_{\mu\lambda} h^{\nu\lambda} \overset{\mu}{P}_i^\mu \overset{k}{P}_\nu^k = \overset{\mu}{P}_i^\mu \overset{k}{P}_\nu^k \delta_\mu^\nu = \delta_i^k. \square$$

THEOREM 5.2. *Denote $\mathcal{A}$ and $\mathfrak{a}$ the determinants of $\overset{i}{P}_\lambda$ and P_i^ν , then*

$$(5.7) \quad \mathcal{A}\mathfrak{a} = 1$$

$$(5.8a) \quad \mathcal{A} = -\gamma\sqrt{\mathfrak{h}}$$

$$(5.8b) \quad \mathfrak{a} = -\gamma/\sqrt{\mathfrak{h}} \quad \text{where } \gamma = \text{sgn}\mathcal{A}$$

Proof. We have first

$$\mathfrak{a} = \mathfrak{e}_{\omega\mu} P_1^\omega P_2^\mu,$$

$$\mathcal{A} = \mathcal{E}^{\omega\mu} P_\omega^1 P_\mu^2.$$

On the other hand,

$$\text{Det}((h_{ij})) = \mathfrak{a}^2 \mathfrak{h} = -1,$$

$$\text{Det}((h^{ij})) = \mathcal{A}^2 \mathfrak{h}^{-1} = -1. \quad \square$$

THEOREM 5.3. *The nonholonomic components of indicators are*

$$(5.9a) \quad \mathcal{E}^{ij} = 2! \delta_{[12]}^{ij},$$

$$(5.9b) \quad \mathfrak{e}_{ij} = 2! \delta_{ij}^{[12]}.$$

THEOREM 5.4. *For even p , the nonholonomic components of ${}^{(p)}k_\lambda^\nu$ are*

$$(5.10) \quad {}^{(p)}k_x^i = (M)_x^p \delta_x^i$$

so that

$${}^{(p)}k_{xi} = (M)_x^p h_{xi}.$$

Proof. Our starting equation is equivalent to (1.4), i.e.

$$(M)_x^p P_x^\nu = {}^{(p)}k_{\mu x}^\nu P_x^\mu.$$

Multiplying this equation by P_ν^i and from (5.2)b,

$$M_x^p \delta_x^i = {}^{(p)}k_{\mu x}^\nu P_\nu^i P_\mu^i = {}^{(p)}k_x^i \quad \square$$

THEOREM 5.5. For even p , the tensors $h_{\lambda\mu}$, ${}^{(p)}k_{\lambda\mu}$, and ${}^{(p)}k^{\lambda\nu}$ may be expressed as follows:

$$(5.11a) \quad h_{\lambda\mu} = 2P_{1(\lambda}P_{2\mu)} \quad , \quad h^{\lambda\nu} = 2P_1^{(\lambda}P_2^{\nu)}$$

$$(5.11b) \quad {}^{(p)}k_{\lambda\mu} = 2M^p P_{[\lambda}^1 P_{\mu]}^2 \quad , \quad {}^{(p)}k^{\lambda\nu} = 2M^p P_2^{[\lambda} P_1^{\nu]}$$

Proof.

$$h_{\lambda\mu} = h_{ij} P_{\lambda}^i P_{\mu}^j = P_{\lambda}^1 P_{\mu}^2 + P_{\lambda}^2 P_{\mu}^1 = P_{2\lambda} P_{1\mu} + P_{1\lambda} P_{2\mu} = 2P_{2(\lambda} P_{1\mu)}$$

$${}^{(p)}k_{\lambda\mu} = M_x^p h_{xi} = M_x^p 2P_{2(\lambda} P_{1\mu)}$$

in virtue of (5.10) and (5.11)a. □

THEOREM 5.6.

$$(5.12a) \quad P_i^{\nu} = P_{\lambda}^j h_{ji} h^{\lambda\nu}$$

$$(5.12b) \quad P_{\lambda}^j = P_i^{\nu} h^{ij} h_{\lambda\nu}$$

Proof. Put $X_i^{\nu} = P_{\lambda}^j h_{ji} h^{\lambda\nu}$

Multiply this equation by P_{μ}^i . Then

$$X_i^{\nu} P_{\mu}^i = P_{\lambda}^j P_{\mu}^i h_{ji} h^{\lambda\nu} = h_{\lambda\mu} h^{\lambda\nu} = \delta_{\mu}^{\nu}$$

Therefore,

$$X_i^{\nu} = P_i^{\nu} \quad \square$$

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