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NUMERICAL SOLUTIONS FOR MODELS OF LINEAR ELASTICITY USING FIRST-ORDER SYSTEM LEAST SQUARES

Chang-Ock Lee

Abstract. Multigrid method and acceleration by conjugate gradient method for first-order system least squares (FOSLS) using bilinear finite elements are developed for various boundary value problems of planar linear elasticity. They are *two-stage* algorithms that first solve for the displacement flux variable, then for the displacement itself. This paper focuses on solving for the displacement flux variable only. Numerical results show that the convergence is uniform even as the material becomes nearly incompressible. Computations for convergence factors and discretization errors are included. Heuristic arguments to improve the convergences are discussed as well.

1. Introduction

Let Ω be a convex polygon or a $C^{1,1}$-domain in $\mathbb{R}^2$ with boundary Γ. Denote the Lamé constants by μ and λ, where $(\mu, \lambda) \in [\mu_1, \mu_2] \times (0, \infty)$ for fixed positive constants μ_1, and μ_2. Boundary value problems with mixed boundary conditions for planar linear elasticity is given in the form

$$
\begin{aligned}
-\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} &= \mathbf{f} \quad \text{in} \quad \Omega\,, \\
\sum_{j=1}^{2}\sigma_{ij}(\mathbf{u})n_j &= 0 \quad \text{on} \quad \Gamma_T, \quad 1 \le i \le 2\,, \\
\mathbf{u} &= \mathbf{0} \quad \text{on} \quad \Gamma_D\,,
\end{aligned}
\tag{1.1}
$$

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where the symbols Δ, ∇, and $\nabla\cdot$ stand for the Laplacian, gradient, and divergence operators, respectively ($\Delta\mathbf{u}$ is the vector of components Δu_i). In addition, $\mathbf{u} = (u_1, u_2)^t$ denotes the displacement, $\mathbf{f} = (f_1, f_2)^t$ a given body force, and $\mathbf{n} = (n_1, n_2)^t$ the unit outerward normal on the boundary. Furthermore, $\sigma_{ij}(\mathbf{u}) = \lambda(\nabla \cdot \mathbf{u})\delta_{ij} + 2\mu\epsilon_{ij}(\mathbf{u})$ is the *stress*, $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ is the *strain*, and δ_{ij} is the Kronecker delta symbol. The boundary Γ satisfies that $\Gamma = \Gamma_T \cup \Gamma_D$ and $\Gamma_T \cap \Gamma_D = \emptyset$. If $\Gamma_D = \emptyset$, then (1.1) is called the *pure traction problem*, and if $\Gamma_T = \emptyset$, then it is called the *pure displacement problem*.

It is well-known that the standard Galerkin finite element formulations using piecewise linear (P-1) finite elements converge for moderate fixed λ, and as $\lambda \to \infty$, i.e., the elastic material becomes incompressible, it seems not to converge at all [1, 17]. In order to overcome this so called the locking phenomenon, several attempts have been made in [2, 11, 14, 16]. These attempts are usually based on *mixed* formulations that lead to discrete equations that are difficult to solve. Recently, in [7] Cai et. al. proposed first-order system least squares (FOSLS) for the Stokes equations and applied it to the pure displacement problem of planar linear elasticity. In addition, in [8] the FOSLS approach was developed for the pure traction problem. They obtained H^1-ellipticity of the least squares functional and showed that H^1-ellipticity is independent of λ. Since the FOSLS formulation generates a symmetric positive definite system, there are good numerical solvers such as multigrid method and/or preconditioned conjugate gradient (PCG) method. Furthermore H^1-ellipticity property guarantees optimal finite element accuracy and multigrid convergence. Also the fact that H^1-ellipticity is independent of λ gives uniform convergence of multigrid method with respect to λ. More recently [6], two FOSLS methods were applied to various boundary value problems of planar linear elasticity. In the paper, numerical results were focused, including finite element accuracy and multigrid convergence estimates that confirm uniform optimal performance-even as the material tends to the incompressible limit.

In this paper we present a more efficient multigrid method than one in [6] for FOSLS formulations of pure displacement problem and mixed boundary value problem of planar linear elasticity. The algorithms are *two-stage*, in which one solves for the displacement flux variable first.

The displacement components can then be obtained as solutions of two scalar Poisson equations. We concentrate on the implementation of the first stage and test numerically the efficiency of the multigrid and acceleration by the conjugate gradient method (PCG) since there are many good Poisson solvers. Furthermore, we show numerically that the multigrid and PCG convergence is uniform with respect to λ and finite element approximation converges to the solution in scaled norm at rate independent of λ. The theoretical backgrounds for the equations of linear elasticity are discussed in [2, 3, 4, 15]. For multigrid method and PCG method, we refer to [5, 12, 13].

This paper is organized as follows. In Section 2 we explain notation, and spaces on which we would solve the problem. FOSLS approaches are introduced in Section 3. In Section 4, we describe the multigrid algorithm and acceleration by the conjugate gradient method. In Section 5 numerical results are presented and improvements are discussed. In the last section, conclusion and remarks are given.

2. Preliminaries

We use standard notation and definitions for the Sobolev spaces $[H^k(\Omega)]^2$, associated inner products $(\cdot,\cdot)_k$, and respective norms $\|\cdot\|_k$, $k > 0$. The space $[L^2(\Omega)]^2$ is interpreted as $[H^0(\Omega)]^2$, in which case the norm and inner product are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. As usual $H_0^k(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_k$. See [10] for more details of function spaces.

Given

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

the operator ∇ is extended to 2-vectors componentwise:

$$\nabla\mathbf{u} = \begin{pmatrix} \nabla u_1 \\ \nabla u_2 \end{pmatrix}.$$

If $\mathbf{U}_1$ and $\mathbf{U}_2$ are 2-vector functions, then we write the block column vector

$$\mathbf{U} \equiv \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{pmatrix}.$$

If D is an operator on 2-vector functions (e.g., $D = \nabla\cdot$, $\nabla\times$, or $\boldsymbol{\tau}\cdot$), then its extension to block column vectors is defined by

$$D\mathbf{U} = \begin{pmatrix} D\mathbf{U}_1 \\ D\mathbf{U}_2 \end{pmatrix}.$$

Inner products and norms on column vector functions are defined in the natural componentwise way: $\|\mathbf{U}\|^2 = \sum_{i=1}^{2} \|\mathbf{U}_i\|^2$. We introduce the *displacement flux* variable $\mathbf{U} = \nabla\mathbf{u}$, that is,

$$\mathbf{U} = (U_1, U_2, U_3, U_4)^t = (\partial_1 u_1, \partial_2 u_1, \partial_1 u_2, \partial_2 u_2)^t .$$

2.1. The pure displacement problem: $\Gamma_T = \emptyset$

Since μ is bounded, we may set $\mu = 1$ without loss of generality. The problem (1.1) is rewritten in the compact form

$$\begin{aligned} -\nabla\cdot(A\nabla\mathbf{u}) &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on} \quad \Gamma, \end{aligned} \tag{2.1}$$

where

$$A = \begin{pmatrix} \lambda+2 & 0 & 0 & \lambda \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \lambda & 0 & 0 & \lambda+2 \end{pmatrix}.$$

Since the definition of $\mathbf{U}$ implies that $\nabla\times\mathbf{U} = \mathbf{0}$ in Ω, then a system that is equivalent to (2.1) is

$$\begin{aligned} \mathbf{U} - \nabla\mathbf{u} &= \mathbf{0} \quad \text{in} \quad \Omega, \\ -\nabla\cdot A\mathbf{U} &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\times\mathbf{U} &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \boldsymbol{\tau}\cdot\mathbf{U} &= \mathbf{0} \quad \text{on} \quad \Gamma, \end{aligned} \tag{2.2}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2)^t$ denotes the unit positive oriented tangential on the boundary. What is more important than this extended system in practice is the system that involves $\mathbf{U}$ only:

$$\begin{aligned} -\nabla\cdot A\mathbf{U} &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\times\mathbf{U} &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \boldsymbol{\tau}\cdot\mathbf{U} &= \mathbf{0} \quad \text{on} \quad \Gamma. \end{aligned} \tag{2.3}$$

It is shown in [8] that this reduced system is well posed. The displacement can be found by solving

$$\begin{aligned} \nabla \mathbf{u} &= \mathbf{U} \quad \text{in} \quad \Omega\,, \\ \boldsymbol{\tau} \cdot \nabla \mathbf{u} &= \mathbf{0} \quad \text{on} \quad \Gamma\,. \end{aligned}$$

See [8] for details.

We define a solution space for the primitive variables by

$$\mathcal{W} = \{\mathbf{u} \in [H_0^2(\Omega)]^2 : \nabla \cdot (A \nabla \mathbf{u}) \in [L^2(\Omega)]^2\}\,.$$

We thus define the solution space for the new variables by

$$\mathcal{V}_D = \{\mathbf{U} \in [H^1(\Omega)]^4 : \boldsymbol{\tau} \cdot \mathbf{U} = \mathbf{0} \text{ on } \Gamma\}\,.$$

2.2. The mixed boundary value problem: $|\Gamma_T| > 0, |\Gamma_D| > 0$

With $\mu = 1$, the extended system of the problem (1.1) is:

$$\begin{aligned} -\nabla \cdot A\mathbf{U} &= \mathbf{f} \quad \text{in} \quad \Omega\,, \\ \nabla \times \mathbf{U} &= \mathbf{0} \quad \text{in} \quad \Omega\,, \\ \mathbf{n} \cdot A\mathbf{U} &= \mathbf{0} \quad \text{on} \quad \Gamma_T\,, \\ \boldsymbol{\tau} \cdot \mathbf{U} &= \mathbf{0} \quad \text{on} \quad \Gamma_D\,. \end{aligned} \tag{2.4}$$

We define the solution space for the new variables by

$$\mathcal{V}_M = \{\mathbf{U} \in [H^1(\Omega)]^4 : \mathbf{n} \cdot A\mathbf{U} = \mathbf{0} \text{ on } \Gamma_T, \boldsymbol{\tau} \cdot \mathbf{U} = \mathbf{0} \text{ on } \Gamma_D\}\,.$$

3. First-Order System Least Squares

Here we describe a FOSLS formulation for solving (2.3), since (2.4) differs from (2.3) only in the boundary condition.

First, define

$$G_0(\mathbf{U}; \mathbf{f}) = \|\mathbf{f} + \nabla \cdot A\mathbf{U}\|^2 + \|\nabla \times \mathbf{U}\|^2 \quad \text{for } \mathbf{U} \in \mathcal{V}_D\,.$$

We consider the rotation defined by the matrix

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and the space $\tilde{\mathcal{V}} \equiv Q\mathcal{V}_D = \{\mathbf{V} = Q\mathbf{U} : \mathbf{U} \in \mathcal{V}_D\}$. Note that $\mathcal{V}_D = Q\tilde{\mathcal{V}}$ and that each vector $\mathbf{U} \in \mathcal{V}_D$ is of the form $\mathbf{U} = Q\mathbf{V}$, $\mathbf{V} \in \tilde{\mathcal{V}}$. Next, consider the scaling defined by the matrix

$$D = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now let $\mathbf{V} = D^{-1}Q\mathbf{U}$ and

$$B = AQD = \begin{pmatrix} \sqrt{2}(1+\frac{1}{\lambda}) & 0 & 0 & \sqrt{2} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \sqrt{2}(1+\frac{1}{\lambda}) & 0 & 0 & -\sqrt{2} \end{pmatrix}.$$

Then $\nabla \cdot A\mathbf{U}$ becomes $\nabla \cdot AQDD^{-1}Q\mathbf{U} = \nabla \cdot B\mathbf{V}$ and $\nabla \times \mathbf{U}$ becomes $\nabla \times QDD^{-1}Q\mathbf{U} = \nabla \times QD\mathbf{V}$. Define a modified functional

$$G_1(\mathbf{W};\mathbf{f}) = \|\mathbf{f}+\nabla\cdot B\mathbf{W}\|^2+\|\nabla\times QD\mathbf{W}\|^2 \quad \text{for } \mathbf{W} \in D^{-1}Q\mathcal{V}_D\,. \tag{3.1}$$

Now, consider the operators

$$\nabla \cdot A = \begin{pmatrix} (\lambda+2)\partial_1 & \partial_2 & \partial_2 & \lambda\partial_1 \\ \lambda\partial_2 & \partial_1 & \partial_1 & (\lambda+2)\partial_2 \end{pmatrix},$$
$$\nabla\times = \begin{pmatrix} \partial_2 & -\partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -\partial_1 \end{pmatrix}.$$

If we first subtract row 2 of $\nabla\times$ from row 1 of $\nabla \cdot A$, then add row 1 of $\nabla\times$ to row 2 of $\nabla \cdot A$, but leave $\nabla\times$ alone, we obtain the operators

$$\nabla \cdot A_s = \begin{pmatrix} (\lambda+2)\partial_1 & \partial_2 & 0 & (\lambda+1)\partial_1 \\ (\lambda+1)\partial_2 & 0 & \partial_1 & (\lambda+2)\partial_2 \end{pmatrix},$$

and

$$\nabla \cdot B_s = \nabla \cdot A_sQD = \begin{pmatrix} \frac{2\lambda+3}{\sqrt{2}\lambda}\partial_1 & \partial_2 & 0 & \frac{1}{\sqrt{2}}\partial_1 \\ \frac{2\lambda+3}{\sqrt{2}\lambda}\partial_2 & 0 & \partial_1 & -\frac{1}{\sqrt{2}}\partial_2 \end{pmatrix},$$
$$\nabla \times QD = \begin{pmatrix} \frac{1}{\sqrt{2}\lambda}\partial_2 & -\partial_1 & 0 & \frac{1}{\sqrt{2}}\partial_2 \\ -\frac{1}{\sqrt{2}\lambda}\partial_1 & 0 & \partial_2 & \frac{1}{\sqrt{2}}\partial_1 \end{pmatrix}.$$

These lead to the, so called, "Stokes" minimization problem given by

(3.2)
$$G_2(\mathbf{V};\mathbf{f}) = \min\{G_2(\mathbf{W};\mathbf{f}) : \mathbf{W} \in [H^1(\Omega)]^4, \boldsymbol{\tau}\cdot QD\mathbf{W} = \mathbf{0} \text{ on } \Gamma\}\,,$$

where

(3.3)
$$G_2(\mathbf{W};\mathbf{f}) = \|\mathbf{f} + \nabla\cdot B_s\mathbf{W}\|^2 + \|\nabla\times QD\mathbf{W}\|^2 \quad \text{for } \mathbf{W}\in D^{-1}Q\mathcal{V}_D\,.$$

Note that the solution of the problem (3.2) is equal to that of the problem to minimize (3.1). However, when the finite element discretization is applied, (3.2) induces rather simpler algebraic systems than minimizing (3.1) because two elimination steps are performed to get the functional (3.3) from (3.1).

4. Numerical Solvers

We now turn to numerical methods for the approximation of the solution $\mathbf{V}$ of (3.2). We subdivide Ω into a set $\mathcal{T}^k$ of non-overlapping rectangles such that $\Omega = \cup_{T\in\mathcal{T}^k} T$ and no vertex of one rectangle lies on the edge of another. The refined set $\mathcal{T}^{k+1}$ is obtained by connecting the midpoints of the opposite edges of the rectangles in $\mathcal{T}^k$. Letting $h_k := \max_{T\in\mathcal{T}^k} \operatorname{diam} T$, then $h_k = 2h_{k+1}$. We define the finite element space:

$$\begin{aligned}\mathcal{V}_k := \{\mathbf{U} :&\mathbf{U}|_T \text{ is bilinear for all } T\in\mathcal{T}^k,\\ &\mathbf{U} \text{ is continuous on } \Omega, \boldsymbol{\tau}\cdot\mathbf{U} = \mathbf{0} \text{ on } \Gamma\}\,.\end{aligned}$$

Note that $\mathcal{V}_k \subset \mathcal{V}_{k+1} \subset [H^1(\Omega)]^4$. Then the following approximation property holds: there exists a constant C such that, for all $\mathbf{V}\in\mathcal{V}_D$, there exists $\mathbf{V}^h\in\mathcal{V}_k$ such that

$$\|\mathbf{V}_j - \mathbf{V}_j^h\| + h_k\|\mathbf{V}_j - \mathbf{V}_j^h\|_1 \le Ch_k^2\|\mathbf{V}_j\|_2, \qquad j = 1,\dots,4\,.$$

4.1. The multigrid algorithm MG

The multigrid algorithm adopted in this paper is most easily described as a functional minimization process in the present context.

Here we describe a simple two-level procedure that provides the basis for V-cycle, W-cycle, and FMG-cycle algorithms in the usual way. See [5] for more details of multigrid methods. Suppose we are given a current approximation $\mathbf{V}_k \in \mathcal{V}_k$ to the solution of the minimization problem

$$G(\mathbf{V};\mathbf{f}) = \min\{G(\mathbf{W};\mathbf{f}) : \mathbf{W} \in \mathcal{V}\}\,. \tag{4.1}$$

Here, $G = G_2$ and the minimization is to be taken over the corresponding subspace $\mathcal{V}$ of $[H^1(\Omega)]^4$ with appropriate boundary conditions. Then MG on level k consists of the following two steps:

(1) **Relaxation:** If $k = 1$, then we define $\mathbf{V}_k \in \mathcal{V}_k$ to be the solution of

$$G(\mathbf{V}_k;\mathbf{f}) = \min\{G(\mathbf{W}_k;\mathbf{f}) : \mathbf{W}_k \in \mathcal{V}_k\}\,.$$

If $k > 1$, then we perform one nodal block Gauss-Seidel relaxation sweep as follows: Let n_k denote the number of nodes on level k including boundary nodes and, for each $l \in \{1, 2, \dots, n_k\}$, let $\mathcal{V}_k^l$ denote the subspace of $\mathcal{V}_k$ of vector functions $\mathbf{W}_k^l$ that are zero at all nodes except node l; then one relaxation sweep is

For $l = 1, 2, \dots, n_k$ in turn,

Let $\mathbf{V}_k^l \in \mathcal{V}_k^l$ solve

$$G(\mathbf{V}_k + \mathbf{V}_k^l;\mathbf{f}) = \min\{G(\mathbf{V}_k + \mathbf{W}_k^l;\mathbf{f}) : \mathbf{W}_k^l \in \mathcal{V}_k^l\}$$

Set $\mathbf{V}_k \leftarrow \mathbf{V}_k + \mathbf{V}_k^l$.

(2) **Coarsening:** If $k > 1$,

Let $\mathbf{V}_{k-1} \in \mathcal{V}_{k-1}$ solve

$$G(\mathbf{V}_k + \mathbf{V}_{k-1};\mathbf{f}) = \min\{G(\mathbf{V}_k + \mathbf{W}_{k-1};\mathbf{f}) : \mathbf{W}_{k-1} \in \mathcal{V}_{k-1}\}$$

Set $\mathbf{V}_k \leftarrow \mathbf{V}_k + \mathbf{V}_{k-1}$.

For this two-grid scheme to allow recursion, we must allow (4.1) to come from a finer level. We do this by defining $G(\mathbf{W};\mathbf{f}) = G_2(\widetilde{\mathbf{W}} + \mathbf{W};\mathbf{f})$, where $\widetilde{\mathbf{W}}$ is an approximation obtained possibly from a level $\tilde{k} > k$ and $\mathbf{W}$ is perhaps restricted to $\mathcal{V}_k$.

4.2. Preconditioned conjugate gradient method PCG

The preconditioned conjugate gradient method for the problem $SU = F$ has been studied by many authors (see [12, 13]). The important step is to *pre-condition* S with a nonsingular symmetric matrix M.

The iterative scheme, for a given initial iterate U^0, is:

Compute $r^0 = F - SU^0$.

Repeat for each ν, $\nu = 1, 2, \cdots$ until convergence;

Solve $Mz^{\nu-1} = r^{\nu-1}$ for $z^{\nu-1}$

If $\nu = 1$, then $\beta_\nu = 0$

$$\text{else } \beta_\nu = \frac{(r^{\nu-1}, z^{\nu-1})}{(r^{\nu-2}, z^{\nu-2})}$$

$p^\nu = z^{\nu-1} + \beta_\nu p^{\nu-1}$

$$\alpha_\nu = \frac{(r^{\nu-1}, z^{\nu-1})}{(Sp^\nu, p^\nu)}$$

$U^\nu = U^{\nu-1} + \alpha_\nu p^\nu$

$r^\nu = F - SU^\nu$

If S and M are symmetric positive definite matrices, then the error vector e^ν for PCG satisfies

$$(e^\nu, M^{-1}Se^\nu)^{1/2} \le 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^\nu (e^0, M^{-1}Se^0)^{1/2}, \tag{4.2}$$

where κ is the condition number of $M^{-1}S$. The conjugate gradient method eliminates error in the direction of eigenvectors associated with extreme isolated eigenvalues. The effective rate is determined by the condition number of the interior eigenvalues. The equation (4.2) shows that the conjugate gradient method converges faster when the eigenvalues of $M^{-1}S$ are clustered together in the sense that the condition number κ is close to 1, i.e. $M \approx S$. Furthermore, we must be able to solve linear system of the form $Mz^\nu = r^\nu$ easily. In these senses, as an effective preconditioner we choose the multigrid method using V-cycle (1,1)-symmetric smoothings.

5. Numerical Results

In this section we first refer to the MG performance studied in [6], and discuss its improvement for the pure displacement problem. In

TABLE 1. Convergence factors for V(1,0)-MG
for the pure displacement problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.5660	0.7127	0.8229	0.8977
$\lambda = 100$	0.5724	0.7331	0.8383	0.8797
$\lambda = 1000$	0.5728	0.7354	0.8387	0.8767

addition, we discuss the performance of PCG algorithm and measure the discretization error. Next, for the mixed boundary value problem we again refer to the MG performance in [6], and study its improvement and the performance of PCG algorithm, too.

5.1. Pure displacement: numerical results in [6]

To study MG performance, we choose the homogeneous pure displacement problem with $\mu = 1$:

$$\begin{aligned} -\Delta \mathbf{u} - (\lambda + 1)\nabla\nabla \cdot \mathbf{u} = \mathbf{0} \quad &\text{in} \quad \Omega = [0,1]^2\,, \\ \mathbf{u} = \mathbf{0} \quad &\text{on} \quad \Gamma. \end{aligned} \tag{5.1}$$

This problem was also adopted as a model problem in [6].

For concreteness, let the east, west, south, and north boundaries of Γ be denoted by Γ_E, Γ_W, Γ_S, and Γ_N, respectively. The boundary conditions for the unit square are equivalent to

$$\begin{aligned} &\frac{1}{\lambda} V_1 - V_4 = 0, \quad V_2 = 0 \quad \text{on} \quad \Gamma_E \cup \Gamma_W\,, \\ &\frac{1}{\lambda} V_1 + V_4 = 0, \quad V_3 = 0 \quad \text{on} \quad \Gamma_S \cup \Gamma_N\,. \end{aligned} \tag{5.2}$$

Of course, the exact solution is zero, but this allows us to study asymptotic MG convergence factors without concern for truncation error. For this purpose, we use random initial guesses. The discretization uses uniform grids. we assess MG efficiency by measuring the convergence factors, i.e., ratio of successive square roots of functional values $G_2(\mathbf{V}; 0)$. This is appropriate because the least squares functional is equivalent to the square of the H^1 norm of the errors.

TABLE 2. Convergence factors for V(1,1)-MG for the pure displacement problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.5007	0.7372	0.8338	0.8962
$\lambda = 100$	0.4974	0.7408	0.8380	0.8903
$\lambda = 1000$	0.4963	0.7402	0.8370	0.8877

TABLE 3. Convergence factors for V(2,1)-MG for the pure displacement problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.4665	0.7144	0.8262	0.8855
$\lambda = 100$	0.4588	0.7177	0.8273	0.8773
$\lambda = 1000$	0.4572	0.7172	0.8263	0.8728

TABLE 4. Convergence factors for W(1,0)-MG for the pure displacement problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.5660	0.6263	0.6274	0.6592
$\lambda = 100$	0.5724	0.6540	0.6518	0.6675
$\lambda = 1000$	0.5728	0.6569	0.6539	0.6672

Tables 1 through 4 are the numerical results in [6] which depict convergence factors measured after 20 multigrid cycles based on the functional based on G_2 in (3.2). The results show that the convergence factors of MG method are independent of λ. Figure 1 is one in [6], that represents the errors after 20 V(1,0)-MG cycles for $h = \frac{1}{32}$ and $\lambda = 1000$. MG for FOSLS based on G_2 produces errors that have predominantly smooth components except near boundary.

5.2. Pure displacement: the improved multigrid algorithm MG*

FIGURE 1. Errors after 20 V(1,0)-MG cycles for the pure displacement problem

Here, we develop an heuristic argument for eliminating an error mode (it is called the table mode) which is not eliminated by the multigrid processes. Assume that λ is very large and note that the shape of V_1 looks like a table with four legs in Figure 1. Therefore we can approximate V_1 with a function that is constant except near four corners of Ω. The function should be zero at four corners. We approximate V_2 and V_3 with 0 on $\Omega \cup \Gamma$. Due to the boundary conditions, $V_4 \approx 0$ on Γ. Therefore, we can approximate V_4 with an almost flat function that satisfies the boundary conditions.

Now we take a vector function $(p, 0, 0, q)$ defined on the discretized domain of Ω such that:

(1) $p = 0$ in each corner, $p = 1$ everywhere else,
(2) $q = \frac{1}{\lambda}$ on $\Gamma_E \cup \Gamma_W$, $q = -\frac{1}{\lambda}$ on $\Gamma_S \cup \Gamma_N$, $q = 0$ in all corners,

TABLE 5. Convergence factors for V(1,0)-MG* for the pure displacement problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.5663	0.7091	0.7768	0.8127
$\lambda = 100$	0.5727	0.7327	0.8037	0.8337
$\lambda = 1000$	0.5731	0.7351	0.8063	0.8358

TABLE 6. Convergence factors for V(1,1)-MG* for the pure displacement problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.3756	0.5648	0.6706	0.7345
$\lambda = 100$	0.3913	0.5979	0.7129	0.7774
$\lambda = 1000$	0.3928	0.6014	0.7174	0.7818

and $q = 0$ in the interior.

Let us call this function the *table* function. Then this function satisfies the boundary conditions (5.2). One can prove that the convergence factor associated with this table mode is bounded away from 1 as $h \to 0$. We modify MG by relaxing in the direction of the table function after each smoothing cycle on each level. (We call it the *table relaxation*.) The computation cost is less than half of one block nodal smoothing cycle. We regard this sequence as one smoothing iteration and denote this improved algorithm by MG*. We may expect that the table relaxation can eliminate the table mode.

The experiments reported here were run on the same environment as in [6], i.e., in double-precision arithmetic on a SUN Ultra-2 Workstation. Tables 5 – 8 depict the convergence factors, and Figure 2 represents errors after 20 V(1,0)-MG* cycles for $h = \frac{1}{32}$ and $\lambda = 1000$. The observed convergence factors and the shape of errors indicate that MG* eliminates the table mode so that MG* gives better performances than MG. Note that the second mode appears in Figure 2.

5.3. Pure displacement: MG* preconditioned conjugate

TABLE 7. Convergence factors for V(2,1)-MG* for the pure displacement problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.2749	0.4755	0.5851	0.6732
$\lambda = 100$	0.2909	0.5131	0.6376	0.7238
$\lambda = 1000$	0.2925	0.5171	0.6435	0.7291

TABLE 8. Convergence factors for W(1,0)-MG* for the pure displacement problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.5663	0.6239	0.6211	0.6098
$\lambda = 100$	0.5727	0.6525	0.6430	0.6370
$\lambda = 1000$	0.5731	0.6557	0.6455	0.6403

gradient method

Now we take a preconditioned conjugate gradient method MG*-PCG which uses V(1,1,sym)-MG* as a preconditioner, where (1,1,sym) means that we perform one symmetric pre- and post-smoothing step, i.e. {the table relaxation + the block nodal relaxation} before the correction step and {the block nodal relaxation + the table relaxation} after the correction step. In order to get the efficiency of the method, we run MG*-PCG restarted every N steps and run it for 10 cycles. We call it MG*-PCG(N). This gives us the worst case N step reduction.

In Table 9, the geometric mean of convergence factors is the 5-th root of the relative error of MG*-PCG(5). Figure 3 represents the errors after 10 MG*-PCG(5) cycles for $h = \frac{1}{32}$ and $\lambda = 1000$. Note that the errors contain a little oscillatory components and all error are at boundaries. Numerical results confirm that performing MG*-PCG is better than just iterating MG* cycles because the conjugate gradient method eliminates isolated modes.

5.4. Pure displacement: discretization errors

To measure the discretization error, we construct a problem with a

FIGURE 2. Errors after 20 V(1,0)-MG* cycles for the pure displacement problem

TABLE 9. Geometric mean of convergence factors for MG*-PCG(5) for the pure displacement problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$
$\lambda = 10$	0.1896	0.2616	0.3482	0.3912	0.4529
$\lambda = 100$	0.1951	0.3053	0.3826	0.4377	0.4897
$\lambda = 1000$	0.1947	0.3081	0.3870	0.4436	0.4950

known nonzero solution. Let the domain Ω be the unit square. The function

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x(1-x)y^2(1-y)^2 \sin \pi x \\ x^2(1-x)^2y^2(1-y)^2 \cos \pi y \end{pmatrix} .$$

FIGURE 3. Errors after 10 MG*-PCG(5) cycles for the pure displacement problem

satisfies the pure displacement boundary condition of (1.1). Regarding $\mathbf{u}$ as the displacement solution of (2.3), we obtain $\mathbf{V}$ and $\mathbf{f}$. With this $\mathbf{f}$, we find the approximate solution of (3.2) using one MG*-PCG(20) cycle on various levels of discretization.

Tables 10 and 11 show that the discretization (relative) errors are $O(h^2)$ with respect to both L^2 and G_2 functional norms ($G_2^{1/2}$). The predicted bounds for errors are $O(h^2)$ in L^2 and $O(h)$ in $G_2^{1/2}$. Note that $\mathbf{u}$ is smooth for this case, and it shows the super-convergence properties of the finite element spaces. This phenomenon was also observed for the least-squares treatment of the Helmholtz equation in [9].

5.5. Mixed boundary conditions: MG, MG* and MG*-PCG performance

TABLE 10. Discretization error in the L^2 norm and their convergence factors for the pure displacement problem

	$\lambda = 10$	$\lambda = 100$	$\lambda = 1000$
	L^2 norm (factor)	L^2 norm (factor)	L^2 norm (factor)
$h = 1/4$	0.4661	0.4292	0.4248
$h = 1/8$	0.1181 (0.2534)	0.1129 (0.2630)	0.1122 (0.2641)
$h = 1/16$	2.893E-2 (0.2450)	2.800E-2 (0.2480)	2.788E-2 (0.2485)
$h = 1/32$	7.075E-3 (0.2446)	6.849E-3 (0.2446)	6.815E-3 (0.2444)
$h = 1/64$	1.743E-3 (0.2464)	1.682E-3 (0.2456)	1.672E-3 (0.2453)

TABLE 11. Discretization error in the G_2 functional norm and their convergence factors for the pure displacement problem

	$\lambda = 10$	$\lambda = 100$	$\lambda = 1000$
	$G_2^{1/2}$ (factor)	$G_2^{1/2}$ (factor)	$G_2^{1/2}$ (factor)
$h = 1/4$	0.2051	0.2089	0.2093
$h = 1/8$	5.127E-2 (0.2500)	5.268E-2 (0.2522)	5.285E-2 (0.2525)
$h = 1/16$	1.282E-2 (0.2500)	1.323E-2 (0.2511)	1.329E-2 (0.2515)
$h = 1/32$	3.207E-3 (0.2502)	3.315E-3 (0.2506)	3.330E-3 (0.2506)
$h = 1/64$	8.018E-4 (0.2500)	8.294E-4 (0.2502)	8.333E-3 (0.2502)

We take the same model problem in [6]:

$$
\begin{aligned}
-\Delta \mathbf{u} - (\lambda + 1)\nabla\nabla \cdot \mathbf{u} = \mathbf{0} \quad &\text{in} \quad \Omega = [0,1]^2\,, \\
\sum_{j=1}^{2} \sigma_{ij}(\mathbf{u}) n_j = 0 \quad &\text{on} \quad \Gamma_T = \Gamma_E \cup \Gamma_N, \quad 1 \le i \le 2\,. \\
\mathbf{u} = \mathbf{0} \quad &\text{on} \quad \Gamma_D = \Gamma_W \cup \Gamma_S\,.
\end{aligned}
$$

To measure the performances we focus on MG and PCG based on G_2,

TABLE 12. Convergence factors for V(1,0)-MG for the mixed boundary value problem

	$h=\frac{1}{4}$	$h=\frac{1}{8}$	$h=\frac{1}{16}$	$h=\frac{1}{32}$
$\lambda=10$	0.5609	0.6589	0.8086	0.8203
$\lambda=100$	0.5622	0.6369	0.8338	0.8293
$\lambda=1000$	0.5620	0.6470	0.8363	0.8289

TABLE 13. Convergence factors for V(1,1)-MG for the mixed boundary value problem

	$h=\frac{1}{4}$	$h=\frac{1}{8}$	$h=\frac{1}{16}$	$h=\frac{1}{32}$
$\lambda=10$	0.3943	0.6167	0.7119	0.7604
$\lambda=100$	0.4040	0.6503	0.7509	0.8002
$\lambda=1000$	0.4049	0.6538	0.7550	0.8042

again. Then, the minimization problem is given by

(5.3)
$$G_2(\mathbf{V};\mathbf{f}) = \min\{G_2(\mathbf{W};\mathbf{f}) : \mathbf{W}\in[H^1(\Omega)]^4, \mathbf{n}\cdot B\mathbf{W}=\mathbf{0} \text{ on } \Gamma_T, \boldsymbol{\tau}\cdot QD\mathbf{W}=0 \text{ on } \Gamma_D\}\,.$$

For the unit square the boundary conditions in (5.3) are

$$\begin{aligned}
&(1+\frac{1}{\lambda})V_1+V_4=0, \quad V_2+V_3=0 \quad \text{on} \quad \Gamma_E\,,\\
&\frac{1}{\lambda}V_1-V_4=0, \qquad V_2=0 \quad \text{on} \quad \Gamma_W\,,\\
&\frac{1}{\lambda}V_1+V_4=0, \qquad V_3=0 \quad \text{on} \quad \Gamma_S\,,\\
&(1+\frac{1}{\lambda})V_1-V_4=0, \quad V_2+V_3=0 \quad \text{on} \quad \Gamma_N\,.
\end{aligned} \tag{5.4}$$

The numerical results in [6] are given in Tables 12 through 15. They show that the convergence factors of MG method are independent of λ. Figure 4 is one in [6], that represents the errors after 20 V(1,0)-MG

TABLE 14. Convergence factors for V(2,1)-MG for the mixed boundary value problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.2939	0.5376	0.6482	0.6742
$\lambda = 100$	0.3101	0.5742	0.6940	0.7103
$\lambda = 1000$	0.3115	0.5781	0.6990	0.7062

TABLE 15. Convergence factors for W(1,0)-MG for the mixed boundary value problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.5609	0.5925	0.6258	0.6080
$\lambda = 100$	0.5622	0.6318	0.6434	0.6359
$\lambda = 1000$	0.5620	0.6362	0.6452	0.6393

cycles for $h = \frac{1}{32}$ and $\lambda = 1000$. Similarly to the pure displacement problem, MG for FOSLS based on G_2 produces errors that have predominantly smooth components. There is also an error mode which is not effectively eliminated by the multigrid processes.

Now, we develop an heuristic argument to eliminate the error mode that appears in Figure 4. We take a vector function $(p, 0, 0, q)$ defined on the discretized domain of Ω such that:

(1) $p = 0$ in each corner, $p(x, y) = 0$ if $x+y \geq 1$, $p(x, y) = 1-(x+y)$ everywhere else,
(2) $q = \frac{1}{\lambda}p$ on Γ_W, $q = -\frac{1}{\lambda}p$ on Γ_S, $q = 0$ everywhere else.

This function satisfies the boundary conditions (5.4). We modify MG by relaxing in the direction of this function after each smoothing cycle on each level. The computation cost is less than half of one block nodal smoothing cycle. We regard this sequence as one smoothing iteration and denote this improved algorithm by MG*. We may expect that the relaxation step can accelerate the MG performance. Tables 16 – 19 depict the convergence factors, and Figure 5 represents errors after 20 V(1,0)-MG* cycles for $h = \frac{1}{32}$ and $\lambda = 1000$.

FIGURE 4. Errors after 20 V(1,0)-MG cycles for the mixed boundary value problem

TABLE 16. Convergence factors for V(1,0)-MG* for the mixed boundary value problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.5054	0.6442	0.7616	0.7513
$\lambda = 100$	0.5099	0.6802	0.7818	0.7638
$\lambda = 1000$	0.5101	0.6845	0.7845	0.7670

Now we take a preconditioned conjugate gradient method MG*-PCG that uses V(1,1,sym)-MG* as a preconditioner with the same way in the pure displacement problem.

In Table 20, the geometric mean of convergence factors is the 5-

TABLE 17. Convergence factors for V(1,1)-MG* for the mixed boundary value problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.3258	0.5318	0.6507	0.6651
$\lambda = 100$	0.3245	0.5737	0.6834	0.6544
$\lambda = 1000$	0.3240	0.5784	0.6874	0.6562

TABLE 18. Convergence factors for V(2,1)-MG* for the mixed boundary value problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.2779	0.4702	0.5847	0.6266
$\lambda = 100$	0.2615	0.4969	0.6220	0.6632
$\lambda = 1000$	0.2598	0.5002	0.6269	0.6683

TABLE 19. Convergence factors for W(1,0)-MG* for the mixed boundary value problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.5054	0.5857	0.6237	0.6066
$\lambda = 100$	0.5099	0.6207	0.6420	0.6321
$\lambda = 1000$	0.5101	0.6246	0.6440	0.6352

th root of the relative error of MG*-PCG(5). Figure 6 represents the errors after 10 MG*-PCG(5) cycles for $h = \frac{1}{32}$ and $\lambda = 1000$. Note that errors are still at boundaries. Numerical results show that performing MG*-PCG is better than just iterating MG* cycles.

6. Conclusion and Remarks

For the pure displacement problem and the mixed boundary value problem, the performances of the multigrid method and acceleration

FIGURE 5. Errors after 20 V(1,0)-MG* cycles for the mixed boundary value problem

TABLE 20. Geometric mean of convergence factors for MG*-PCG(5) for the mixed boundary value problem

	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
$\lambda = 10$	0.1747	0.2689	0.3321	0.3703
$\lambda = 100$	0.1682	0.2784	0.3512	0.3912
$\lambda = 1000$	0.1690	0.2765	0.3511	0.3953

by the conjugate gradient method were illustrated. For the pure displacement problem, MG*-PCG gives good performance. For the mixed boundary value problem, the convergence factors are between those of the pure traction and pure displacement problems. For a substantial

FIGURE 6. Errors after 10 MG*-PCG(5) cycles for the mixed boundary value problem

Γ_D, MG*-PCG appears to be good.

In the pure displacement problem, the table mode seems to come from the treatment of corners in the unit square. We noticed that a similar situation happened for the mixed boundary value problem. If we design new FOSLS formulation and/or numerical strategy which can eliminate the table mode, then we could get better multigrid performance and/or the performance of the preconditioned conjugate gradient method.

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Department of Mathematics

Inha University
Inchon 402-751, Korea
E-mail: colee@math.inha.ac.kr