

TOTAL CURVATURE FOR SOME MINIMAL SURFACES

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ABSTRACT. In this paper, we estimate the total curvature of non-parametric minimal surfaces by using the properties of univalent harmonic mappings defined on $\Delta = \{z : |z| > 1\}$.

1. Introduction

Hengartner and Schober [3] studied the class Σ of all complex-valued, harmonic, orientation-preserving, univalent mappings f defined on $\Delta = \{z : |z| > 1\}$ that are normalized at infinity by $f(\infty) = \infty$. Such functions admit the representation

$$(1.1) \quad f(z) = h(z) + \overline{g(z)} + A \log|z|,$$

where

$$h(z) = \alpha z + \sum_{k=0}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in Δ and $0 \leq |\beta| < |\alpha|$. In addition, $a = \overline{f_z}/f_z$ is analytic and satisfies $|a(z)| < 1$.

Now let Ω be a doubly connected domain in the extended w -plane having the point $w = \infty$ as one of its boundary continua. Let S be a nonparametric surface over Ω given by

$$S = \{(u, v, \varphi(u, v)) : u + iv \in \Omega\}.$$

We shall associate with every solution $\varphi(u, v)$ of the classical equation of minimal surfaces,

$$(1.2) \quad (1 + \varphi_v^2)\varphi_{uu} - 2\varphi_u\varphi_v\varphi_{uv} + (1 + \varphi_u^2)\varphi_{vv} = 0$$

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and the functions

$$\psi(u, v) = \int \frac{\varphi_u dv - \varphi_v du}{\sqrt{1 + \varphi_u^2 + \varphi_v^2}};$$

$$F = \varphi + i\psi;$$

$$\omega = \frac{\varphi_u - i\varphi_v}{1 + \sqrt{1 + \varphi_u^2 + \varphi_v^2}}.$$

Note that the integral defining ψ is path-independent by virtue of (1.2) and that ψ is determined but for an additive constant. Also, we have $|\omega| < 1$. The functions F and ω are to be considered as defined on S . While ω is single-valued on this surface, F may be multi-valued, but its branches differ only by constants.

If $\varphi(u, v)$ is interpreted as the potential of a flow of a hypothetical “Chaplygin gas” whose density ρ and speed q are connected by the relation

$$\rho^2(1 + q^2) = 1,$$

then ψ is the stream-function, F the complex potential, and $\varphi_u - i\varphi_v$ the conjugate complex velocity (cf.[1]).

From now on, assume that φ is nonconstant; that is, S is not a horizontal plane. Then S is a minimal surface if and only if S admits a conformal reparametrization of the form

$$(1.3) \quad S = \{(\operatorname{Re}G_1(z), \operatorname{Re}G_2(z), \operatorname{Re}F(z)) : z = x + iy \in \Delta\},$$

where

$$G_1(z) = \frac{1}{2} \int F' \left(\frac{1}{\omega} - \omega \right) dz, \quad G_2(z) = \frac{-i}{2} \int F' \left(\frac{1}{\omega} + \omega \right) dz.$$

The function

$$f(z) = \operatorname{Re}G_1(z) + i\operatorname{Re}G_2(z) = \frac{1}{2} \int \frac{F'(z)}{\omega(z)} dz - \frac{1}{2} \overline{\int \omega(z) F'(z) dz}$$

is a univalent harmonic mapping from Δ onto Ω with $f(\infty) = \infty$ and $\overline{f_z}/f_z = -\omega^2$. In Δ , the variables F and ω considered as functions of z are regular analytic, dF/dz and ω are single-valued, and

$(1/\omega)(dF/dz) \neq 0, \infty$. The function $\omega(z)$ is regular at $z = \infty$ and $|\omega(\infty)| < 1$. Furthermore, $F'(z)/\omega(z)$ is regular and different from zero at ∞ . Also observe that we may assume f is orientation-preserving and that we may obtain any other set of isothermal parameters by applying a conformal mapping to Δ (cf.[1]). Since $f(z) \in \Sigma$, f has the representation (1.1).

Note that $\omega(u, v) = 0$ if and only if the unit normal vector to the surface

$$\vec{n} = (-\varphi_u, -\varphi_v, 1)/\sqrt{1 + \varphi_u^2 + \varphi_v^2}$$

is $(0, 0, 1)$ at $(u, v, \varphi(u, v))$. This means that S has a horizontal tangent plane at points where ω vanishes.

In this paper, we estimate the total curvature of nonparametric minimal surfaces over Ω by using the properties of univalent harmonic mappings f in Δ with $f(\infty) = \infty$.

2. Curvature

Let S be a nonparametric minimal surface over Ω , then S is given as in (1.3). The first fundamental form for the Euclidean length on S is

$$ds^2 = \lambda^2 |dz|^2 \quad \text{where} \quad \lambda = |h' + \frac{A}{2z}| + |g' + \frac{\bar{A}}{2z}|.$$

Therefore the Gaussian curvature K of S is

$$K = \frac{-\Delta \log \lambda}{\lambda^2} = \frac{-|(g' + \frac{\bar{A}}{2z})'(h' + \frac{A}{2z}) - (g' + \frac{\bar{A}}{2z})(h' + \frac{A}{2z})'|^2}{|(h' + \frac{A}{2z})(g' + \frac{\bar{A}}{2z})| \left(|h' + \frac{A}{2z}| + |g' + \frac{\bar{A}}{2z}| \right)^4}.$$

Since $a = \bar{f}_z/f_z = (g' + \frac{\bar{A}}{2z})/(h' + \frac{A}{2z}) = -\omega^2$, we obtain

$$K = \frac{-4|\omega'|^2}{\left(|h' + \frac{A}{2z}| + |g' + \frac{\bar{A}}{2z}| \right)^2 (1 + |\omega|^2)^2}.$$

If we now let D be a domain whose closure is in Δ , the surface over $f[D](\subset \Omega)$ has total curvature given by

$$(2.1) \quad T = \iint_D K dA = \iint_D K \lambda^2 dx dy \quad \text{where} \quad z = x + iy.$$

By substituting λ^2 and K into (2.1), we have that

$$T = - \iint_D \frac{4|\omega'|^2}{(|\omega|^2 + 1)^2} dxdy.$$

The estimate $|\omega'|/(1 - |\omega|^2) \leq 1/(|z|^2 - 1)$ from Schwarz's lemma for $|z| > 1$ implies

$$|T| = 4 \iint_D \frac{|\omega'|^2}{(|\omega|^2 + 1)^2} dxdy \leq 4 \iint_D \left(\frac{1 - |\omega|^2}{1 + |\omega|^2} \right)^2 \frac{1}{(|z|^2 - 1)^2} dxdy.$$

Since $\left(\frac{1 - |\omega|^2}{1 + |\omega|^2} \right)^2 \leq 1$ for $|\omega| < 1$, we obtain

$$|T| \leq \iint_D \frac{4}{(|z|^2 - 1)^2} dxdy.$$

Consider the surface

$$S_{r_1}^{r_2} = \{(\operatorname{Re} G_1(z), \operatorname{Re} G_2(z), \operatorname{Re} F(z)) : z = x + iy \in D_{r_1}^{r_2}\}$$

where $D_{r_1}^{r_2} = \{z : 1 < r_1 \leq |z| \leq r_2\}$. Then the estimate of $|T|$ for $S_{r_1}^{r_2}$ is given by

$$|T| \leq \frac{4\pi(r_2^2 - r_1^2)}{(r_1^2 - 1)(r_2^2 - 1)}.$$

We have proved the following theorem.

THEOREM. *Let S be a nonparametric minimal surface over a doubly connected domain Ω . Then an estimate for the total curvature T of $S_{r_1}^{r_2}$ is*

$$|T| \leq \frac{4\pi(r_2^2 - r_1^2)}{(r_1^2 - 1)(r_2^2 - 1)}.$$

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