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NOTES ON MAXIMAL COMMUTATIVE SUBALGEBRAS OF 14 BY 14 MATRICES

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Abstract. Let Ω be the set of all commutative k-subalgebras of 14 by 14 matrices over a field k whose dimension is 13 and index of Jacobson radical is 3. Then we will find the equivalent condition for a commutative subalgebra to be maximal .

1. Introduction

In this paper, k will denote an arbitrary field. We will denote $M_{n\times n}(k)$ by T_n. All k-algebras will be assumed to contain a (multiplicative) identity $1 \neq 0$. Let R be a local commutative k-subalgebra of T_n. R is called a maximal commutative k-subalgebra of T_n if R satisfies the following property : If R' is a commutative, k-subalgebra of T_n and $R \subseteq R'$, then $R = R'$. Thus, a maximal, commutative, k-subalgebra of T_n is a maximal element with respect to inclusion in the set of all maximal, commutative, k-subalgebras of T_n. We will denote the set of all maximal, commutative, k-subalgebra of T_n by $\mathcal{M}_n(k)$.

Thus, if $C_{T_n}(S) = \{A \in T_n \mid As = sA,\ for\ all\ s \in S\}$ is the centralizer of a set S in T_n, then $R \in \mathcal{M}_n(k)$ if and only if $C_{T_n}(R) = R$.

We will use the notation $(R, J(R), k) \in \mathcal{M}_n(k)$ to denote a local, commutative, k-algebra $R \in \mathcal{M}_n(k)$ which has $J(R)$ as its Jacobson radical and k as its residue class field. Let $i(J(R))$ be the index of nilpotency of the ideal $J((R))$ and let $\Omega = \{(R, J(R), k) \in \mathcal{M}_{14}(k) \mid dim_k(R) = 13, i(J(R)) = 3\}$.

In [3], R.C. Courter constructed an algebra $\mathcal{C} \in \mathcal{M}_{14}(k)$ which is local, $dim_k(\mathcal{C}) = 13$, and $i(J(\mathcal{C})) = 3$. It has been conjectured for a long time that the set Ω has only one isomorphism class $[\mathcal{C}]$. It was

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proved in [4] that the isomorphism class $[\mathcal{C}]$ in Ω is not unique. It is very natural to ask how many nonisomorphic classes are in Ω.

In section 2, we will recall some general properties of $(R, J(R), k) \in \Omega$.

In section 3, we will prove an equivalent condition for a commutative subalgebra R of $M_{14}(k)$ having dimension 13 and $i(J(R)) = 3$.

2. Algebras in Ω

If $(R, J(R), k) \in \Omega$, then we have the following properties. The proofs can be found in [4].

THEOREM 2.1. *Suppose that $(R, J(R), k) \in \Omega$. Then there exists $(R_1, J(R_1), k) \in \Omega$ such that R and R_1 are conjugate and each element $r \in J(R_1)$ is a matrix of the following form*

$$\begin{pmatrix} O_2 & O & O \\ A & O_{10} & O \\ C & B & O_2 \end{pmatrix}. \tag{1}$$

Here, O_n denotes the zero matrix of size n by n, $A \in M_{10\times 2}(k), B \in M_{2\times 10}(k)$, and $C \in T_2$.

Recall that the socle of R, $Soc(R) = Ann_R(J(R)) = \{r \in R \mid rJ(R) = (0)\}$.

LEMMA 2.2. *Let R and R_1 be finite dimensional, commutative, k-algebras. If $R \cong R_1$ as k-algebras, then $Soc(R) \cong Soc(R_1)$.*

THEOREM 2.3. *Suppose $(R, J, k) \in \Omega$. Then, $dim_k(Soc(R)) = 4$. Furthermore, R is conjugate to an $(R_1, J(R_1), k) \in \Omega$ such that each element of $Soc(R_1)$ has the following form.*

$$r = \begin{pmatrix} O_2 & O & O \\ O & O_{10} & O \\ C & O & O_2 \end{pmatrix}. \tag{2}$$

Thus, we can always assume that a specific representative R of an isomorphism class $[R]$ has the following property. Every element $r \in$

$J(R)$ can be written in the form

$$r = \begin{pmatrix} O_2 & O & O \\ A & O_{10} & O \\ C & B & O_2 \end{pmatrix}.$$

Furthermore, the socle of R is the set of all matrices of the form

$$Soc(R) = \left\{ \begin{pmatrix} O_2 & O & O \\ O & O_{10} & O \\ C & O & O_2 \end{pmatrix} \mid C \in T_2 \right\}.$$

3. Main Results

If $(R, J(R), k) \in \Omega$, then by Theorem 2.1, we may assume every $r \in J(R)$ has the form in (1). From Theorem 2.3, we may assume every $r \in Soc(R)$ has the form in (2). We can then write $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$, where

$$\lambda_i = \begin{pmatrix} O_2 & O & O \\ A_i & O_{10} & O \\ O & B_i & O_2 \end{pmatrix}, \check{E}_{mn} = \begin{pmatrix} O_2 & O & O \\ O & O_{10} & O \\ E_{mn} & O & O_2 \end{pmatrix} \tag{3}$$

Here, E_{mn} is the (i,j)-th matrix unit in T_2. Conversely, suppose R is a commutative, k-subalgebra of T_{14} of the form $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$, where $dim_k R = 13$ and $\lambda_1, \ldots, \lambda_8$ have the form given in (3). Then, R is a local ring with Jacobson radical given by $J(R) = (\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22})$ and residue class field k. We will give a necessary and sufficient condition on the A_i's and B_i's which will imply $R \in \Omega$.

For a matrix $A \in M_{m\times n}(k)$, we let $ker\ A = \{u \in M_{1\times m}(k) | uA = 0\}$ and $NS(A) = \{v \in M_{n\times 1}(k) | Av = 0\}$.

THEOREM 3.1:. *Let $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$ be a commutative, k-subalgebra of T_{14}. We assume $dim_k R = 13$ and each λ_i has the form given in (3). Suppose $\bigcap_{i=1}^{8} ker(A_i) = (0)$ and $\bigcap_{i=1}^{8} NS(B_i) = (0)$. If $r \in C_{T_{14}}(R)$, then r has the following form.*

$$r = \begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ Z & Q & O_2 \end{pmatrix} + aI_{14}, \quad a \in k.$$

Proof. Let $r = \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix} \in C_{T_{14}}(R)$. Here, $X_1, X_9 \in T_2$ and $X_5 \in T_{10}$. Then $r\check{E}_{ij} = \check{E}_{ij}\ r$ and

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix} \begin{pmatrix} O_2 & O & O \\ A_i & O_{10} & O \\ W & B_i & O_2 \end{pmatrix} = \begin{pmatrix} O_2 & O & O \\ A_i & O_{10} & O \\ W & B_i & O_2 \end{pmatrix} \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix}$$

for all $i = 1, \ldots, 8$. Thus, we have the following equations.

(a) $$X_2A_i + X_3W = 0$$

(b) $$X_3B_i = 0$$

(c) $$X_5A_i + X_6W = A_iX_1$$

(d) $$X_6B_i = A_iX_2$$

(e) $$A_iX_3 = 0$$

(f) $$X_8A_i + X_9W = WX_1 + B_iX_4$$

(g) $$X_9B_i = WX_2 + B_iX_5$$

(h) $$WX_3 + B_iX_6 = 0$$

These equations hold for all $i = 1, \ldots, 8$ and all $W \in T_2$. We also have the equations obtained by replacing A_i and B_i in (a) through (h) with the zero matrix. Since $X_3W = 0$ for all $W \in T_2$, we have $X_3 = 0$. Then, (a) implies $X_2A_i = 0$ for all $i = 1, \ldots, 8$. Thus, $X_2 \in \bigcap_{i=1}^{8} ker(A_i) = (0)$. Hence, $X_2 = 0$. Equation (h) implies $B_iX_6 = 0$ for all $i = 1, \ldots, 8$. Thus, $X_6 \in \bigcap_{i=1}^{8} NS(B_i) = (0)$. Hence, $X_6 = 0$. Since $X_9W = WX_1$ for all $W \in T_2$ from (f), we have the following equations.

$$X_9E_{11} = E_{11}X_1, X_9E_{12} = E_{12}X_1$$
$$X_9E_{21} = E_{21}X_1, X_9E_{22} = E_{22}X_1.$$

Let $X_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $X_9 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Here, $a_{ij}, b_{ij} \in k$, $i, j = 1, 2$. Then, $a_{11} = a_{22} = b_{11} = b_{22}$ and $a_{12} = a_{21} = b_{12} = b_{21} = 0$. Thus, $X_1 = X_9 = a_{11}I_2$. In (c), let $W = 0$. Then, $X_5A_i = A_iX_1 = A_i(a_{11}I_2) = a_{11}A_i$. Hence, $(X_5 - a_{11}I_{10})A_i = 0$, for all $i = 1, \ldots, 8$. Thus, $X_5 - a_{11}I_{10} \in \bigcap_{i=1}^{8} ker(A_i) = (0)$ which implies $X_5 = a_{11}I_{10}$. Therefore, the proof is completed. □

In the next theorem, we characterize those P's and Q's for which $r \in C_{T_{14}}(R)$.

THEOREM 3.2. *Let* $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$ *be the k-subalgebra in Theorem 3.1. Let* $r = \begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ Z & Q & O_2 \end{pmatrix} + aI_{14} \in T_{14}$. *Then,* $r \in C_{T_{14}}(R)$ *if and only if* $\begin{pmatrix} (Row_1Q)^T \\ Col_1P \\ (Row_2Q)^T \\ Col_2P \end{pmatrix} \in NS(\Lambda)$. *Here,* Row_iQ *is the i-th row of* Q, Col_iP *is the i-th column of* P, *and* $\Lambda \in M_{32\times 40}(k)$ *is the following matrix.*

$$(4) \quad \Lambda = \begin{bmatrix} \begin{pmatrix} (Col_1A_1)^T & -Row_1B_1 & O & O \\ (Col_2A_1)^T & O & O & -Row_1B_1 \\ O & -Row_2B_1 & (Col_1A_1)^T & O \\ O & O & (Col_2A_1)^T & -Row_2B_1 \end{pmatrix} \\ \vdots \\ \vdots \\ \vdots \\ \begin{pmatrix} (Col_1A_8)^T & -Row_1B_8 & O & O \\ (Col_2A_8)^T & O & O & -Row_1B_8 \\ O & -Row_2B_8 & (Col_1A_8)^T & O \\ O & O & (Col_2A_8)^T & -Row_2B_8 \end{pmatrix} \end{bmatrix}$$

Proof. Suppose $r \in C_{T_{14}}(R)$. Then, for all $i = 1, \ldots, 8$,

$$\begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ Z & Q & O_2 \end{pmatrix} \begin{pmatrix} O_2 & O & O \\ A_i & O_{10} & O \\ W & B_i & O \end{pmatrix} = \begin{pmatrix} O_2 & O & O \\ A_i & O_{10} & O \\ W & B_i & O \end{pmatrix} \begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ Z & Q & O_2 \end{pmatrix}.$$

Therefore, $QA_i = B_iP$ for $i = 1, \ldots, 8$. Let

$$A_i = \begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} \\ \vdots & \vdots \\ a_{101}^{(i)} & a_{102}^{(i)} \end{pmatrix}, B_i = \begin{pmatrix} b_{11}^{(i)} & \cdots & b_{110}^{(i)} \\ b_{21}^{(i)} & \cdots & b_{210}^{(i)} \end{pmatrix}$$

for $i = 1, \ldots, 8$

$$P = \begin{pmatrix} p_{11} & p_{12} \\ \vdots & \vdots \\ p_{101} & p_{102} \end{pmatrix}, Q = \begin{pmatrix} q_{11} & \cdots & q_{110} \\ q_{21} & \cdots & q_{210} \end{pmatrix}.$$

Here, $a_{mn}^{(i)},\ b_{nm}^{(i)},\ p_{mn},\ q_{nm} \in k,\ n = 1, 2, m = 1, \ldots, 10$. Since $QA_i = B_iP$ for all $i = 1, \ldots, 8$, we have

(5)

$$\sum_{j=1}^{10} q_{1j}a_{j1}^{(i)} - \sum_{j=1}^{10} b_{1j}^{(i)}p_{j1} = 0, \qquad \sum_{j=1}^{10} q_{1j}a_{j2}^{(i)} - \sum_{j=1}^{10} b_{1j}^{(i)}p_{j2} = 0$$
$$\sum_{j=1}^{10} q_{2j}a_{j1}^{(i)} - \sum_{j=1}^{10} b_{2j}^{(i)}p_{j1} = 0, \qquad \sum_{j=1}^{10} q_{2j}a_{j2}^{(i)} - \sum_{j=1}^{10} b_{2j}^{(i)}p_{j2} = 0.$$

It is easy to check (5) is equivalent to

$$\Lambda \begin{pmatrix} (Row_1Q)^T \\ Col_1P \\ (Row_2Q)^T \\ Col_2P \end{pmatrix} = 0. \tag{6}$$

Conversely, if P and Q satisfy Equation (6), then $QA_i = B_iP$ for all $i = 1, \ldots, 8$. Hence, $r \in C_{T_{14}}(R)$. □

THEOREM 3.3. *Let* $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$ *be a commutative, k-subalgebra of* T_{14}. *We assume* $dim_k R = 13$ *and each* λ_i *has the form given in (3). Then, the following two statements are equivalent.*

(a) $R \in \Omega$

(b) $\bigcap_{i=1}^{8} ker(A_i) = (0)$, $\bigcap_{i=1}^{8} NS(B_i) = (0)$, *and* $rank(\Lambda) = 32$.

In Theorem 3.3, Λ is the 32×40 matrix given in (4).

Proof. (a) $\Rightarrow$ (b) Let $u = (u_1, \ldots, u_{10}) \in \bigcap_{i=1}^{8} ker(A_i)$. Then,

$$\begin{pmatrix} O_2 & O & O \\ O & O_{10} & O \\ O & \binom{u}{o} & O_2 \end{pmatrix} \in Soc(R).$$

Theorem 2.3 implies $u = (0)$ and hence $\bigcap_{i=1}^{8} ker(A_i) = (0)$.
Let $v = (v_1, \ldots, v_{10})^T \in \bigcap_{i=1}^{8} NS(B_i)$. Then,

$$\begin{pmatrix} O_2 & O & O \\ (vo) & O_{10} & O \\ O & O & O_2 \end{pmatrix} \in Soc(R).$$

Since $Soc(R) = L(\check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22})$, $v = (0)$. This implies that, $\bigcap_{i=1}^{8} NS(B_i) = (0)$. Let

$$\alpha_i = \begin{pmatrix} (Row_1 B_i)^T \\ Col_1 A_i \\ (Row_2 B_i)^T \\ Col_2 A_i \end{pmatrix}, \quad i = 1, \ldots, 8. \tag{7}$$

Since $\lambda_i \in R = C_{T_{14}}(R)$, $\alpha_i \in NS(\Lambda)$ by Theorem 3.2. Since $\lambda_1, \ldots, \lambda_8$ are linearly independent, $\alpha_1, \ldots, \alpha_8$ are linearly independent. Hence, $dim_k NS(\Lambda) \geq 8$. Let $w \in NS(\Lambda)$. Since $w \in M_{40\times 1}(k)$, we can write w as follows.

$$w = \begin{pmatrix} (Row_1 Q)^T \\ Col_1 P \\ (Row_2 Q)^T \\ Col_2 P \end{pmatrix}$$

for some $P \in M_{10\times 2}(k)$ and $Q \in M_{2\times 10}(k)$. Let

$$r = \begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ O & Q & O_2 \end{pmatrix}.$$

Then, by Theorem 3.2, $r \in C_{T_{14}}(R) = R$. Thus, $r = c_1\lambda_1 + \cdots + c_8\lambda_8$ for some $c_i \in k,\ i = 1, \ldots, 8$. Hence, $w = c_1\alpha_1 + \cdots + c_8\alpha_8$. Therefore, $dim_k NS(\Lambda) \leq 8$ and hence $dim_k NS(\Lambda) = 8$. We conclude $rank(\Lambda) = 32$.

(b) $\Rightarrow$ (a) Since $rank(\Lambda) = 32$, $dim_k NS(\Lambda) = 8$. Let α_i, be the vectors defined in (7). Since $dim_k R = 13$, $\lambda_1, \ldots, \lambda_8$ are linearly independent over k. It easily follows that $\alpha_1, \ldots, \alpha_8$ are linearly independent over k. Thus, $\{\alpha_1, \ldots, \alpha_8\}$ is a basis of $NS(\Lambda)$. If $r \in C_{T_{14}}(R)$, then Theorem 3.1 and Theorem 3.2 imply

$$\begin{pmatrix} (Row_1 Q)^T \\ Col_1 P \\ (Row_2 Q)^T \\ Col_2 P \end{pmatrix} \in NS(\Lambda).$$

Thus,

$$\begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ O & Q & O_2 \end{pmatrix} \in L(\lambda_1, \ldots, \lambda_8).$$

Therefore, $r \in R$ and hence $C_{T_{14}}(R) = R$. We conclude $R \in \mathcal{M}_{14}(k)$. □

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