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ON M-SETS

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Abstract. In this paper, we introduce the subsupratopology and the productsupratopology. In particular, we will show the class induced by a restricted supratopology contains the restricted class induced by the supratopology.

1. Introduction

Let X, Y and Z be topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of X. The closure of A is denoted by A^-, and the interior of A is denoted by A^0. In 1963, Levine[1] defined a subset A of X to be a semi-open set if $A \subset A^{0-}$. The complement of a semi-open set is called semi-closed. The family of all semi-open sets in X will be denoted by $SO(X)$. In 1965, Njastad[6] defined a subset A in space X to be an α-set if $A \subset A^{0-0}$. The complement of an α-set is called α-closed. The family of all α-sets in X will be denoted by $\alpha(X)$. Clearly all open sets are α-sets in topological spaces. The class $\alpha(X)$ of all α-sets is a topology finer than the given topology on X [6,Proposition 2]. Also $\alpha(X)$ consists of exactly those sets A for which $A \cap B \in SO(X)$ for all $B \in SO(X)$ [6,Proposition 1]. These properties will be extended to the case of our new nearly open sets instead of α-set. A subset A of a space X is said to be a pre-open set if $A \subset A^{-0}$ in [2]. The complement of a pre-open set is called the pre-closed set. The family of all pre-open sets in X will be denoted by $PO(X)$. In 1983, Mashhour et al.[3] called a subclass τ^* of power set of X a supratopology if τ^* contains X and it is closed under arbitrary union. Let (X, τ) be a topological space and τ^* be a supratopology on X. We call τ^* a supratopology associated

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with τ if $\tau \subset \tau^*$. In case that the supratopology τ^* is associated with τ, the supratopology τ^* is denoted by (X, τ, τ^*).

Let (X, τ^*) be a supratopological space. A subset A of X is called an m-set with τ^* if $A \cap T \in \tau^*$ for all $T \in \tau^*$.

The class of all m-sets with τ^* is denoted by τ_m. In [5], we proved that the class τ_m of all m-sets with τ^* is a topology on X.

2. The m-sets induced by $(\tau^*)_Y$

We will show that the class of elements of a supratopology restricted on a subset Y of X is also a supratopology on Y.

THEOREM 2.1. *Let (X, τ^*) be a supratopological space and Y be a subset of X. Then the class $(\tau^*)_Y := \{A \cap Y | A \in \tau^*\}$ is a supratopology on Y.*

Proof. Since X is an element of τ^*, $Y = X \cap Y$ is in $(\tau^*)_Y$. For $A_i \cap Y \in (\tau^*)_Y$, $\cup(A_i \cap Y) = (\cup A_i) \cap Y \in (\tau^*)_Y$. Thus the class $(\tau^*)_Y$ is closed under arbitrary union. Therefore, the class $(\tau^*)_Y$ is a supratopology on Y. $\square$

We call $(Y, (\tau^*)_Y)$ the subsupratopological space on Y. The members of $(\tau^*)_Y$ are called subsupraopen subsets of Y. The class of all m-sets with $(\tau^*)_Y$ is denoted by $\tau_Y m$. Indeed the class $\tau_Y m$ of all m-sets with $(\tau^*)_Y$ is a topology on Y. We will show the class induced by a restricted supratopology contains the restricted class induced by the supratopology. The following theorem is very useful in the sequel.

THEOREM 2.2. *Let (X, τ^*) be a supratopological space and Y be a subset of X. Let $\tau_m(Y) := \{U \cap Y | U \in \tau_m\}$. Then $\tau_m(Y) \subset \tau_Y m$.*

Proof. Let $W \in \tau_m(Y)$. There exists an m-set U such that $W = U \cap Y$. Any $A \cap Y \in (\tau^*)_Y$ for $A \in \tau^*$, $W \cap (A \cap Y) = (U \cap Y) \cap (A \cap Y) = (U \cap A) \cap Y \in (\tau^*)_Y$, because $U \in \tau_m, A \in \tau^*$ and $U \cap A \in \tau^*$. Thus $W \in \tau_Y m$. Therefore $\tau_m(Y) \subset \tau_Y m$. $\square$

But we get the following example which shows $\tau_m(Y) \neq \tau_Y m$ in general.

EXAMPLE 2.3. Let the supratopology $\tau^* = \{\emptyset, X, \{a\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{b,c,d\}, \{a,b,d\}\}$ on $X = \{a,b,c,d\}$. Then we get the subsupratopology $(\tau^*)_Y = \{\emptyset, Y, \{a\}, \{b,c\}, \{b\}, \{a,b\}\}$ on $Y = \{a,b,c\}$. Then the class $\tau_Y m = \{\emptyset, Y, \{a\}, \{b,c\}, \{b\}, \{a,b\}\}$ is induced by $(\tau^*)_Y$. Since the class $\tau_m = \{\emptyset, X, \{a\}, \{b,c,d\}\}$ is induced by τ^*, we can find the class $\tau_m(Y) = \{\emptyset, Y, \{a\}, \{b,c\}\}$. Thus the class $\tau_m(Y)$ does not contain $\tau_Y m$.

In other words, $\tau_Y m$ need not be a relative topology of Y in (X, τ_m) in Example 2.3.

COROLLARY 2.4. *Let $(X, \tau, SO(X))$ be a supratopological space. And Y be a pre-open subset of X. Then $\alpha_Y(X) =: \{U \cap Y | U \in \alpha(X)\}$ is a subclass of $\alpha(Y)$.*

Proof. Since $\tau^* = SO(X)$, $\tau_m = \alpha(X)$. Thus $\alpha_Y(X) = \{U \cap Y | U \in \alpha(X),\ Y$ is a pre-open set $\}$. By [4, Lemma 1.1], we get the result. □

COROLLARY 2.5. *Let $(X, \tau, SO(X))$ be a supratopological space and let Y be a subset of X. Then $\alpha_Y(X) = \{U \cap Y | U \in \alpha(X)\}$ is a subclass of $\tau_Y m$ which is induced by $SO_Y(X) =: \{V \cap Y | V \in SO(X)\}$.*

Proof. Since $\tau^* = SO(X)$, $\tau_m = \alpha(X)$. Then $\alpha_Y(X) = \{U \cap Y | U \in \alpha(X)\}$. By Theorem 2.2, we get the result. □

COROLLARY 2.6. *Let (X, τ^*) be a supratopological space and Y be a subset of X. Then Y is an m-set with $(\tau^*)_Y$.*

Proof. Since X is an m-set with τ^*, $X \cap Y$ is an m-set with $(\tau^*)_Y$ by Theorem 2.2. Thus Y is an m-set with $(\tau^*)_Y$. □

THEOREM 2.7. *Let (X, τ^*) be a supratopological space and Y be a subset of X. If Y is an m-set with τ^* and A is an m-set with $(\tau^*)_Y$. Then A is an m-set with τ^*.*

Proof. Let Y be an m-set with τ^* and let A be an m-set with $(\tau^*)_Y$ and $T \in \tau^*$. Then $A \cap T = A \cap T \cap Y$. Since A is an m-set with $(\tau^*)_Y$, thus $A \cap T \cap Y \in (\tau^*)_Y$. By the definition of $(\tau^*)_Y$, there is a supraopen U, such that $A \cap T \cap Y = U \cap Y$. Since Y is an m-set with τ^*, $U \cap Y \in \tau^*$. Therefore, $A \cap T \in \tau^*$. □

The assumption that Y is an m-set can not be replaced with a supraopen set.

EXAMPLE 2.8. Let $X = \{a, b, c, d\}$ and $\tau^* = \{\emptyset, X, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ be a supratopology on X. Then $\tau_m = \{\emptyset, X, \{a, b, c\}\}$. Let $Y = \{a, b\}$ be a subset of X. Then $Y \in \tau^*$ and $(\tau^*)_Y = \{\emptyset, Y, \{b\}\}$. Thus $\tau_Y m = \{\emptyset, Y, \{b\}\}$. the set $\{b\}$ is an m-set with $(\tau^*)_Y$ and $Y \in \tau^*$, but $\{b\}$ is not an m-set with τ^*.

COROLLARY 2.9. *Let $(X, \tau, SO(X))$ be a supratopological space. Let Y be an α-set in X and an m-set A be induced by $SO_Y(X)$. Then A is an α-set in X.*

Proof. Since Y is an α-set in X, Y is an m-set with $SO(X)$. Since $SO_Y(X) = (\tau^*)_Y$ and A is an m-set with $SO_Y(X)$. By Theorem 2.7, A is an m-set with $SO(X)$. That is, A is an α-set in X. $\square$

section3. The m-sets induced by $(\tau \times \mu)^*$

Let (X, τ^*) and (Y, μ^*) be supratopological spaces. We will construct a product supra topology on $X \times Y$ which is associated with the product topology on $X \times Y$. Let $(\tau \times \mu)^* = \{\cup U_\alpha | U_\alpha = A_\alpha \times B_\alpha,\ A_\alpha \in \tau^*$ and $B_\alpha \in \mu^*\}$. Then it is closed under arbitrary union and contains $X \times Y$.

Let (X, τ^*) and (Y, μ^*) be supratopological spaces. We recall that [3] a subset A of the product space $X \times Y$ is an S-closed subset in $X \times Y$ if for each $(x, y) \in (X \times Y) - A$, there exist two supraopen sets U and V which contain x and y respectively, such that $(U \times V) \cap A = \emptyset$.

THEOREM 3.1. *Let (X, τ^*) and (Y, μ^*) be supratopological spaces. Then the class $(\tau \times \mu)^*$ consists of exactly the complement of an S-closed subsets in $X \times Y$.*

Proof. Let A be an S-closed subset in $X \times Y$. For each $(x, y) \in (X \times Y) - A$, there exist two supraopen sets $U(x)$ and $V(y)$ which contain x and y respectively, such that $(U(x) \times V(y)) \cap A = \emptyset$. Thus $(x, y) \in (U(x) \times V(y)) \subset (X \times Y) - A$. Therefore, $\cup_{(x,y)\in(X\times Y)-A}(U(x) \times V(y)) = (X \times Y) - A = A^c$, That is, A^c is an element of $(\tau \times \mu)^*$.

Let $\cup U_\alpha$ be an element of $(\tau \times \mu)^*$. We will show that $(\cup U_\alpha)^c$ is an S-closed subset in $X \times Y$. Since $(X \times Y) - (\cup U_\alpha)^c = \cup U_\alpha$. For each $(x, y) \in \cup U_\alpha$, there exists U_α with $U_\alpha = A_\alpha \times B_\alpha$ and $A_\alpha \in \tau^*$

and $B_\alpha \in \mu^*$ such that $(x, y) \in U_\alpha = (A_\alpha \times B_\alpha) \subset \cup U_\alpha$. That is, $(A_\alpha \times B_\alpha) \cap (\cup U_\alpha)^c = \emptyset$. And $(\cup U_\alpha)^c$ is an S-closed subset in $X \times Y$.□

By the above theorem, the S-closed subset in $X \times Y$ equals the complement of a supraopen subset with supratopology $(\tau \times \mu)^*$. We know that the class $(\tau \times \mu)^*$ is a supratopology associated with product topology on $X \times Y$. The class of all m-sets with $(\tau \times \mu)^*$ is denoted by $(\tau \times \mu)_m$. The product topology of $X \times Y$ induced by (X, τ_m) and (Y, μ_m) is denoted by $\tau_m \times \mu_m$.

THEOREM 3.2. *Let (X, τ^*) and (Y, μ^*) be supratopological spaces. The class $\tau_m \times \mu_m$ is contained in the class $(\tau \times \mu)_m$ of all m-sets induced by $(\tau \times \mu)^*$.*

Proof. Let $(A \times B) \in (\tau_m \times \mu_m)$. For any $\cup U_\alpha \in (\tau \times \mu)^*$, $(A \times B) \cap (\cup U_\alpha) = \cup((A \times B) \cap U_\alpha)$. Since $(A \times B) \cap U_\alpha$ is a supraopen set in $(\tau \times \mu)^*$, $\cup((A \times B) \cap U_\alpha)$ is a supraopen set in $(\tau \times \mu)^*$. Therefore, $(A \times B) \in (\tau \times \mu)_m$. □

DEFINITION 3.3. Let (X, τ^*) and (Y, μ^*) be supratopological spaces. A subset A of $X \times Y$ is an M-closed set in $X \times Y$, if for each $(x, y) \in X \times Y - A$, there exist two m-sets U with τ^* and V with μ^* which contain x and y respectively, such that $(U \times V) \cap A = \emptyset$.

The following theorem shows that if a subset is an M-closed set in $\tau_m \times \mu_m$ then it is an m-closed set in $(\tau \times \mu)_m$.

THEOREM 3.4. *Let (X, τ^*) and (Y, μ^*) be supratopological spaces. The complement of an M-closed set in $X \times Y$ is an element of the class $(\tau \times \mu)_m$.*

Proof. Let A be an M-closed set in $X \times Y$. We will show that A^c belongs to the class $(\tau \times \mu)_m$. For each $(x, y) \in (X \times Y) - A$, there exist two m-sets $U(x) \in \tau_m$ and $V(y) \in \mu_m$ which contain x and y respectively, such that $(x, y) \in (U(x) \times V(y)) \subset (X \times Y) - A$. Thus $\cup_{(x,y)\in(X\times Y)-A}(U(x) \times V(y)) = (X \times Y) - A$. Therefore A^c is an m-set. □

Let (X, τ) and (Y, μ) be two topological spaces. We recall that a subset A of $X \times Y$ is an α-closed set in the product space $X \times Y$, if for each $(x, y) \in X \times Y - A$, there exist two α-sets U and V which contain x and y respectively, such that $(U \times V) \cap A = \emptyset$.

We investigate the relations of closed, α-closed, m-closed and S-closed in the following theorem.

THEOREM 3.5. *Let (X, τ, τ^*), (Y, μ, μ^*) be supratopological spaces. Let A be a subset of $(X \times Y, (\tau \times \mu)^*)$.*

(1) In case $\tau \subset \tau_m$ and $\mu \subset \mu_m$, if A is closed, then A is also m-closed.

(2) In case $\tau^* = SO(X)$ and $\mu^* = SO(Y)$, A is m-closed if and only if A is α-closed.

(3) In case $\tau^* = PO(X)$ and $\mu^* = PO(Y)$, if A is α-closed, then A is m-closed.

Proof. The proof is obvious. □

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