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THE DENSITY FOR JUMP PROCESSES IN CANONICAL STOCHASTIC DIFFERENTIAL EQUATION

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Abstract. The existence of density of process, which is given by canonical stochastic differential equation, can be proved by the Picard's method([5]) also.

I. Introduction

In this paper, we study the existence of density of law of process given by canonical stochastic differential equation(SDE). Since J.B.Bismut studied the density for the jump-type process, R.Léandre, P.Malliavin (c.f. [1]) and others studied it for various jump-type processes by particular methods. In this work, we use Picard's method([5]) mainly to study the existence of smooth density of process given by the canonical SDE whose driving process is a jump-type Lévy process.

Let Lévy measure ν satisfy (1) and (2);

(1). $|\frac{\lambda^+}{\lambda^-}| < \infty$ as $\rho \to 0$, where λ^+ and λ^- are the largest and the smallest eigenvalues of $V(\rho)$, respectively, where

$$V(\rho) = \int_{|z|\le\rho} zz^*\nu(dz), \quad \rho \in (0,1).$$

(2). For some $\alpha \in (0,2)$, $\liminf_{\rho\to 0} \rho^{-\alpha} v(\rho) > 0$, where

$$v(\rho) = \int_{|z|\le\rho} |z|^2\nu(dz).$$

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Consider a solution of canonical SDE of the form;

$$d\xi_t(x) = \sum_{\alpha=1}^{m} V_\alpha(\xi_t(x)) \diamond dZ^\alpha(t),$$

where $V_1, V_2, \cdots, V_m$ are the smooth complete vector fields on $\mathbf{R}^d$, driven by an $\mathbf{R}^m$-valued Lévy process $\{Z(t); t \geq 0\}$ with Lévy measure ν defined by

$$Z(t) = bt + \int_0^t \int_{|z|\leq 1} z\tilde{N}_p(ds, dz) + \int_0^t \int_{|z|>1} zN_p(ds, dz).$$

Then, under some conditions, we can get a process $\{\xi_t(x); 0 \leq r \leq t \leq T\}$;

$$\begin{aligned} \xi_t(x) &= x + \sum_{\alpha=1}^{m} \int_0^t b^\alpha V_\alpha(\xi_r(x))dr + \sum_{\alpha=1}^{m} \int_0^t V_\alpha(\xi_{r-}(x))dZ_d^\alpha(r) \\ &\quad + \sum_{0<r\leq t} c(\xi_{r-}(x), \Delta Z(r)), \qquad (*) \end{aligned}$$

where

$$c(x, z) = \exp(\sum_{\alpha=1}^{m} z^\alpha V_\alpha)(x) - x - \sum_{\alpha=1}^{m} z^\alpha V_\alpha(x).$$

In general, the time parameters of processes are given by subscripts, but in some special case(for example, $Z(t)$), they are given by normal letters as in above equation$(*)$.

For vector fields $V_\alpha, \alpha = 1, 2, \cdots, m$ of SDE$(*)$, we put

$$\begin{aligned} \mathbf{E}_1 &:= \{V_1, V_2, \cdots, V_m\}, \\ & \quad \cdots\cdots\cdots, \\ \mathbf{E}_{l+1} &:= [\mathbf{E}_l, (V_1, V_2, \cdots, V_m)], \end{aligned}$$

where $[\cdot, \cdot]$ is the Lie bracket. Then we get the result; suppose that

$$Vect(\cup_{i=1}^\infty \mathbf{E}_i(x)) = \mathbf{R}^d,$$

then, under the Conditions (1) and (2), the law of $\xi_t(x)$ of SDE$(*)$ has a C_b^∞-density for all $0 < t \leq T$.

In Section II, we introduce the canonical SDE and the stochastic flow as the solution of canonical SDE under some conditions. Furthermore, we introduce the result which is given by Picard's method. In section III, we prove the result. To prove the result by Picard's method, we need some calculations for vector fields to get the regularity for $\xi_t(x)$ of SDE$(*)$ etc.

II. Canonical SDE and result

Let $(\tilde{\Omega}, \mathcal{F}, P)$ be a probability space where the filtration $\{\mathcal{F}_t; t \in [0, \infty)\}$ of the right continuous increasing family of sub-σ-fields of $\mathcal{F}$ is defined. Let $\{X_t(x); t \geq 0\}$ be an C-valued semi-martingale equipped with the characteristic (α, β, μ), and $\{K_t, t \geq 0\}$ be a positive predictable process satisfying

$$\int_0^T K_t dA_t < \infty \quad a.s. \quad for \quad any \quad T > 0,$$

for an integrable predictable increasing process A_t.

Condition $(A^{m+\delta})$ (1). $\alpha(x, y, t)$ is a predictable continuous $\tilde{C}_b^{m+1+\delta}$-valued process satisfying

$$\|\alpha(t)\|_{m+1+\delta} \leq K_t \quad a.s.,$$

(2). $\beta(x, t)$ is a predictable continuous $C_b^{m+\delta}$-valued process satisfying

$$\|\beta(t)\|_{m+\delta} \leq K_t \quad a.s.,$$

(3). The measure μ_t is supported by $C_b^{m+1+\delta}$. Further, there exists a Borel set $U \subset C_b^{m+1+\delta}$ such that for some constant $C > 0$, $\|v\|_{m+1+\delta} \leq C$ for all $v \in U$, and

$$\mu_t(U^c) \leq K_t, \qquad \int_U \|v\|^2_{m+1+\delta} \mu_t(dv) \leq K_t.$$

Consider a canonical SDE of the form;

$$d\xi_t(x) = \sum_{\alpha=1}^{m} V_\alpha(\xi_t(x)) \diamond dZ^\alpha(t) \qquad (II-1)$$

driven by a vector field-valued Lévy process

$$X_t(x) = \sum_{\alpha=1}^{m} V_\alpha(x) Z^\alpha(t),$$

where $Z(t) = (Z^1(t), Z^2(t), \cdots, Z^m(t))$ is an $\mathbf{R}^m$-valued Lévy process and $V_1, V_2, \cdots, V_m$ are the smooth complete vector fields on $\mathbf{R}^d$ given by the form;

$$V_\alpha = \sum_{i=1}^{d} v_\alpha^i(x)\frac{\partial}{\partial x^i}, \quad \alpha = 1, 2, \cdots, m.$$

We assume that $v_\alpha^i(x)$ are C^∞-functions with bounded derivatives of all orders ≥ 1. By the solution of canonical SDE (II-1), under the Condition $(A^{m+\delta})$, we can define an $\mathbf{R}^d$-valued stochastic flows of diffeomorphisms $\{\xi_{s,t}(x); 0 \leq s \leq r \leq t \leq T\}$ adapted to $\mathcal{F}_t = \sigma(Z(s); s \leq t)$ satisfying ;

$$\begin{aligned} \xi_{s,t}(x) &= x + \sum_{\alpha=1}^{m}\int_s^t V_\alpha(\xi_{s,r}(x)) \diamond dZ^\alpha(r) \\ &= x + \sum_{\alpha=1}^{m}\int_s^t V_\alpha(\xi_{s,r}(x)) \circ dZ_c^\alpha(r) + \sum_{\alpha=1}^{m}\int_s^t V_\alpha(\xi_{s,r-}(x))dZ_d^\alpha(r) \\ &\quad + \sum_{0<s\leq r\leq t} c(\xi_{s,r-}(x), \Delta Z(r)), \qquad (II-2) \end{aligned}$$

where

$$c(x,z) = \exp(\sum_{\alpha=1}^{m} z^\alpha V_\alpha)(x) - x - \sum_{\alpha=1}^{m} z^\alpha V_\alpha(x).$$

Let $\Omega \subset \tilde{\Omega}$ be the set of all integer-valued measures on $\mathbf{R}_+ \times \mathbf{R}^m$. For each $u = (t, z) \in [0, T] \times \mathbf{R}^m$, we define a transformation ε_u^+ on Ω by

$$\begin{aligned} \varepsilon_u^+ N_p(A) &= \varepsilon_u^- N_p(A) + I_A(u), \\ \varepsilon_u^- N_p(A) &= N_p(A \cap \{u\}^c). \end{aligned}$$

For a functional F defined on Ω, we define also an operator D by

$$D_u F = F \circ \varepsilon_u^+ - F.$$

If $\tau = (u_1, u_2, \cdots, u_k)$, they are defined by

$$\varepsilon_\tau^+ = \varepsilon_{u_1}^+ \circ \varepsilon_{u_2}^+ \circ \cdots \circ \varepsilon_{u_k}^+,$$

and

$$D_\tau = D_{u_1} \cdots D_{u_k}.$$

In the case $k = 0$, we use the convention $\varepsilon_\phi^+\omega = \omega$, and $D_\phi F = F$.

Suppose that $\hat{N}_p$ is the product of the Lebesgue measure on $\mathbf{R}_+$ and that a Lévy measure ν on $\mathbf{R}^m$. Then random variables on $\tilde{\Omega}$ are functionals of the Lévy process $\{Z(t); t \geq 0\}$ with Lévy measure ν defined by

$$Z(t) = bt + \int_0^t \int_{|z|\le 1} z\tilde{N}_p(ds,dz) + \int_0^t \int_{|z|>1} zN_p(ds,dz), \qquad (II-3)$$

where b and z are the elements of $\mathbf{R}^m$, and the compensator $\hat{N}_p(ds,dz)$ of Poisson random measure N_p is of the form;

$$\hat{N}_p(ds,dz) = \nu(dz)ds.$$

To get the existence of a smooth density, we need some sufficient conditions which are applied to the case of SDE (II-7).

Condition (B). Lévy measure ν satisfies;

(1). $|\frac{\lambda^+}{\lambda^-}| < \infty$ as $\rho \to 0$, where λ^+ and λ^- are the largest and the smallest eigenvalues of $V(\rho)$, respectively, where

$$V(\rho) = \int_{|z|\le\rho} zz^*\nu(dz), \quad \rho \in (0,1).$$

(2). For some $\alpha \in (0,2)$, $\liminf_{\rho\to 0} \rho^{-\alpha} v(\rho) > 0$, where

$$v(\rho) = \int_{|z|\le\rho} |z|^2 \nu(dz).$$

Now, we introduce a Proposition for a random variable F, which is in [5];

PROPOSITION II-1. *Suppose that the Lévy measure ν satisfies the Condition (B). Let $t > 0$, and let F be an $\mathbf{R}^d$-valued functional of Lévy process satisfying the following (1) and (2);*

(1), for any p and k,

$$\|\text{ess sup}\{\frac{|D_\tau F|}{\prod_{j=1}^k |z_j|}; \tau = ((r_1,z_1),\cdots,(r_k,z_k)), |z_j| \le 1\}\|_p < \infty, \quad (II-4)$$

(2), there exists a matrix-valued process Ψ_r such that for $|z| \le 1, p \ge 1$,

$$\|D_{r,z}F - \Psi_r z\|_p \le C_p |z|^q, \qquad (II-5)$$

for some $q > 1$, and

$$\|(\det \int_0^t \Psi_r \Psi_r^* dr)^{-1}\|_p < \infty. \qquad (II-6)$$

Then F has an C_b^∞-density.

For the SDE (II-1), we consider a Lévy process $Z(t) = (Z^1(t), Z^2(t), \cdots, Z^m(t))$ of the type (II-3) whose component forms are following;

$$\begin{aligned} Z^\alpha(t) &= b^\alpha t + \int_0^{t+} \int_{\mathbf{R}^m \setminus \{0\}} z^\alpha I_{\{|z| \leq 1\}} \tilde{N}_p(ds, dz) \\ &\quad + \int_0^{t+} \int_{\mathbf{R}^m \setminus \{0\}} z^\alpha I_{\{|z|>1\}} N_p(ds, dz), \quad \alpha = 1, 2, \cdots, m. \end{aligned}$$

Then, under the (3) of Condition $(A^{m+\delta})$ for Lévy measure ν of (II-3), we can get a stochastic flow of diffeomorphims of the form; for all $0 \leq s \leq r \leq t \leq T$,

$$\begin{aligned} \xi_{s,t}(x) &= x + \sum_{\alpha=1}^m \int_s^t b^\alpha V_\alpha(\xi_{s,r}(x)) dr + \sum_{\alpha=1}^m \int_s^t V_\alpha(\xi_{s,r-}(x)) dZ_d^\alpha(r) \\ &\quad + \sum_{0<s\leq r\leq t} c(\xi_{s,r-}(x), \Delta Z(r)), \qquad (II-7) \end{aligned}$$

where

$$c(x, z) = \exp(\sum_{\alpha=1}^m z^\alpha V_\alpha)(x) - x - \sum_{\alpha=1}^m z^\alpha V_\alpha(x).$$

In particular, if $s = 0$ is fixed, then we get a process by setting; $\xi_{0,t}(x) := \xi_t(x)$, and we know also that SDE (II-1) has a unique solution $\{\xi_t(x); 0 \leq t \leq T\}$ satisfying (II-7).

To get that the Jacobian matrix of $\exp(\sum_\alpha z^\alpha V_\alpha)(x)$ is invertible, we introduce a Proposition;

PROPOSITION II-2. *A matrix linear differential equation of the form;*

$$\begin{aligned} \frac{d}{dt} X_t &= A(t) X_t, \\ X_0 &= I \qquad\qquad (II-8) \end{aligned}$$

has a solution X_t and $det(X_t) \neq 0$.

LEMMA II-1. *the Jacobian matrix of* $\exp(\sum_\alpha z^\alpha V_\alpha)(x)$ *is invertible a.s.*

Proof. If we put

$$\varphi_t(x, z) = \exp(t \sum_\alpha z^\alpha V_\alpha)(x),$$

then we get

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\partial}{\partial x^i}\varphi_t(x,z)\Big) &= \frac{\partial}{\partial x^i}\Big(\frac{d}{dt}\varphi_t(x,z)\Big) \\ &= \sum_{\alpha=1}^{m} z^\alpha \sum_{j=1}^{d} \frac{\partial}{\partial x^j} V_\alpha(\varphi_t(x,z)) \frac{\partial}{\partial x^i}\varphi_t^j(x,z),\end{aligned}$$

where $\frac{\partial}{\partial x^i}\varphi_t^j(x,z)$ is an $d\times d$-matrix. Therefore, if we think equation;

$$\begin{aligned}\frac{d}{dt}D_x\varphi_t(x,z) &= A(t)D_x\varphi_t(x,z), \\ D_x\varphi_0(x,z) &= I, \qquad\qquad (II-9)\end{aligned}$$

where $A(t)$ is an $d\times d$-matrix, we know that the differential equation (II-9) is a matrix linear differential equation. Thus, from the Proposition II-2, we see that $D_x\tilde{c}(x,z)$ is invertible a.s. $\square$

On the other hand, for the flow $\{\xi_{s,t}(x); 0\le s\le r\le t\}$ of equation (II-7), we can get the Jacobian matrix $\nabla\xi_{s,t}(x)$ at x as following (c.f. (6-30) and (10-16) of [1], and [2]);

$$\begin{aligned}\nabla\xi_{s,t}(x) &= I + \sum_{\alpha=1}^{m}\int_s^t b^\alpha \nabla V_\alpha(\xi_{s,r}(x))\nabla\xi_{s,r}(x)dr \\ &+ \sum_{\alpha=1}^{m}\int_s^t \nabla V_\alpha(\xi_{s,r-}(x))\nabla\xi_{s,r-}(x)dZ_d^\alpha(r) \\ &+ \sum_{0<s\le r\le t} \nabla c(\xi_{s,r-}(x),\Delta Z_r)\nabla\xi_{s,r-}(x), \qquad (II-10)\end{aligned}$$

where

$$\nabla c(x,z) = \nabla\exp\Big(\sum_{\alpha=1}^{m} z^\alpha V_\alpha\Big)(x) - I - \sum_{\alpha=1}^{m} z^\alpha\nabla V_\alpha(x).$$

Now, for vector fields $V_\alpha, \alpha=1,2,\cdots,m$, we put

$$\begin{aligned}\mathbf{E}_1 &:= \{V_1, V_2, \cdots, V_m\}, \\ &\cdots\cdots\cdots, \\ \mathbf{E}_{l+1} &:= [\mathbf{E}_l, (V_1, V_2, \cdots, V_m)],\end{aligned}$$

where $[\cdot,\cdot]$ is the Lie bracket. Then we get the result.

THEOREM. *Suppose that*

$$Vect(\cup_{i=1}^{\infty}\mathbf{E}_i(x)) = \mathbf{R}^d. \qquad (II-11)$$

Then, under the (3) of Condition $(A^{m+\delta})$ *for Lévy measure* ν *of (II-3) and Condition (B), the law of* $\xi_t(x)$ *in (II-7) has an* C_b^{∞}*-density for all* $0 \le t \le T$.

III. Proof of the theorem

Put $\Psi_r := \nabla\xi_{r,t}(\xi_r)\tilde{\mathbf{V}}(\xi_r)$, where $\tilde{\mathbf{V}}(x)$ is the matrix in the vector $\mathbf{V} = (V_1, V_2, \cdots, V_m)$ of vector fields such that

$$\mathbf{V} = \begin{pmatrix} v_1^1(x) & v_1^2(x) & \cdots & v_1^d(x) \\ v_2^1(x) & v_2^2(x) & \cdots & v_2^d(x) \\ & \cdots & & \\ & \cdots & & \\ v_m^1(x) & v_m^2(x) & \cdots & v_m^d(x) \end{pmatrix} \begin{pmatrix} \partial/\partial x^1 \\ \partial/\partial x^2 \\ \cdot \\ \cdot \\ \partial/\partial x^d \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{V} &= \tilde{\mathbf{V}}(x)(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^d})^* \\ &= (\tilde{\mathbf{V}}^1(x), \tilde{\mathbf{V}}^2(x), \cdots, \tilde{\mathbf{V}}^d(x))(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^d})^*. \end{aligned}$$

The proof of the theorem can be got by followings;

PROPOSITION III-1. *Assumption (II-11) implies (II-6);*

$$\|(\det \int_s^t \Psi_r \Psi_r^* dr)^{-1}\|_p < \infty,$$

i.e., as a Wiener functional, the random variable F *of Proposition II-1 is non-degenerate in the sense of Malliavin.*

To get the proof of this Proposition, we need some Lemmas.

LEMMA III-1. (c.f.[3]) *We get that $\nabla\xi_t(x)$ is invertible, and get*

$$(\nabla\xi_{s,t}(x))^{-1} = I - \sum_{\alpha=1}^{m}\int_s^t (\nabla\xi_{s,r}(x))^{-1} b^\alpha \nabla V_\alpha(\xi_{s,r}(x))dr$$

$$+ \sum_{\alpha=1}^{m}\int_s^t (\nabla\xi_{s,r-}(x))^{-1}\nabla V_\alpha(\xi_{s,r-}(x))dZ_d^\alpha(r)$$

$$+ \sum_{0<s\le r\le t}(\nabla\xi_{s,r-}(x))^{-1}\nabla c^{-1}(\xi_{s,r-}(x),\Delta Z(r)), \quad (III-1)$$

where

$$\nabla c^{-1}(x,z) = (\nabla\exp(\sum_{\alpha=1}^{m} z^\alpha V_\alpha)(x))^{-1} - I + \sum_{\alpha=1}^{m} z^\alpha \nabla V_\alpha(x).$$

Proof. Since $\xi_{s,t}(x)$ satisfies SDE (II-7), its Jacobian matrix $\nabla\xi_{s,r}(x)$ satisfies (II-8), and $\nabla c^{-1}(x,z)$ is defined by Lemma II-1. We consider the linear SDE for unknown matrix-valued process $X_{s,t}$;

$$X_{s,t} = I - \sum_{\alpha=1}^{m}\int_s^t X_{s,r} b^\alpha \nabla V_\alpha(\xi_{s,r})dr + \sum_{\alpha=1}^{m}\int_s^t X_{s,r-}\nabla V_\alpha(\xi_{s,r-})dZ_d^\alpha(r)$$

$$+ \sum_{0<s\le r\le t} X_{s,r-}\nabla c^{-1}(\xi_{s,r-},\Delta Z(r)).$$

It has an unique solution $X_{s,t}$. Further, we can show directly that the product $X_{s,t}\nabla\xi_{s,t}$ satisfies

$$d_t(X_{s,t}\nabla\xi_{s,t}) = d_t X_{s,t}\cdot\nabla\xi_{s,t}(x) + X_{s,t}\cdot d_t\nabla\xi_{s,t}(x) = 0.$$

Therefore, $X_{s,t}\nabla\xi_{s,t}(x) = I$ holds a.s.. Thus $\nabla\xi_{s,t}(x)$ is invertible and the inverse $(\nabla\xi_{s,t}(x))^{-1}$ satisfies equation (III-1). □

LEMMA III-2. (c.f.[3]) *For $0\le s\le t\le T$ and a given vector field V, we get;*

$$(\nabla\xi_t(x_0))^{-1}V(\xi_t(x_0)) = V(x_0) - \sum_{\alpha=1}^{m}\int_0^t (\nabla\xi_s)^{-1} b^\alpha [V,V_\alpha](\xi_s)ds$$

$$-\sum_{k=1}^{m}\int_0^t (\nabla\xi_{s-})^{-1}(\nabla\exp(\cdot))^{-1}[V,V_k](\exp(\cdot))(\xi_{s-})dZ_d^k(s) \qquad (III-2)$$

$$-\sum_{k=1}^{m}\sum_{0<s\le t}(\nabla\xi_{s-})^{-1}\{(\nabla\exp(\cdot))^{-1}[V,V_k](\exp(\cdot))(\xi_{s-}) - [V,V_k](\xi_{s-})\}\Delta Z_s^k,$$

where $\exp(\cdot) = \exp(\sum_{\alpha=1}^{m} \Delta Z^{\alpha} V_{\alpha})$.

Proof. In view of Ito's formula for semi-martingale with jumps, we have

$$\begin{aligned} V(\xi_t) &= V(x_0) + \sum_{\alpha=1}^{m} \int_0^t \nabla V(\xi_s) b^{\alpha} V_{\alpha}(\xi_s) ds \\ &+ \sum_{\alpha=1}^{m} \int_0^t \nabla V(\xi_{s-}) V_{\alpha}(\xi_{s-}) dZ_d^{\alpha}(s) \\ &+ \sum_{0<s\le t} [V(\exp(\cdot)(\xi_{s-})) - V(\xi_{s-}) - \sum_{k=1}^{m} \Delta Z_s^k \nabla V(\xi_{s-}) V_k(\xi_{s-})]. \end{aligned}$$

Now, for the product of two semi-martingales $X_t = (\nabla \xi_t)^{-1}$ and $Y_t = V(\xi_t)$, we have the formula

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \int_0^t X_s \circ dY_c(s) + \int_0^t (\circ dX_c(s)) Y_s \\ &+ \int_0^t X_{s-} dY_d(s) + \int_0^t dX_s Y_d(s-) + [X_d(t), Y_d(t)], \end{aligned}$$

where

$$\begin{aligned} X_c(t) &= -\sum_{\alpha=1}^{m} \int_0^t (\nabla \xi_s)^{-1} b^{\alpha} \nabla V_{\alpha}(\xi_s) ds, \\ Y_c(t) &= \sum_{\alpha=1}^{m} \int_0^t \nabla V(\xi_s) b^{\alpha} V_{\alpha}(\xi_s) ds, \\ X_d(t) &= \sum_{\alpha=1}^{m} \int_0^t (\nabla \xi_{s-})^{-1} \nabla V_{\alpha}(\xi_{s-}) dZ_d^{\alpha}(s) + \sum_{0<s\le t} (\nabla \xi_{s-})^{-1} [(\nabla \exp(\cdot)(\xi_{s-}))^{-1} \\ &\quad - I + \sum_{k=1}^{m} \Delta Z_s^k \nabla V(\xi_{s-}) V_k(\xi_{s-})], \\ Y_d(t) &= \sum_{\alpha=1}^{m} \int_0^t \nabla V(\xi_{s-}) V_{\alpha}(\xi_{s-}) dZ_d^{\alpha}(s) \\ &\quad + \sum_{0<s\le t} [V(\exp(\cdot)(\xi_{s-})) - V(\xi_{s-}) - \sum_{k=1}^{m} \Delta Z_s^k \nabla V(\xi_{s-}) V_k(\xi_{s-})]. \end{aligned}$$

We have also

$$\int_0^t X_s \circ dY_c(s) = \sum_{\alpha=1}^m \int_0^t (\nabla \xi_s)^{-1} \nabla V(\xi_s) b^\alpha V_\alpha(\xi_s) ds,$$

and

$$\int_0^t \circ dX_c(s) Y_s = -\sum_{\alpha=1}^m \int_0^t (\nabla \xi_s)^{-1} b^\alpha \nabla V_\alpha(\xi_s) V(\xi_s) ds.$$

Since $[V_\alpha, V] = \nabla V V_\alpha - \nabla V_\alpha V$, we have

$$\int_0^t X_s \circ dY_c(s) + \int_0^t \circ dX_c(s) Y_s = -\sum_{\alpha=1}^m \int_0^t (\nabla \xi_s)^{-1} b^\alpha [V, V_\alpha](\xi_s) ds.$$

On the other hand, a direct computation yields

$$\begin{aligned}
&\int_0^t X_{s-} dY_d(s) + \int_0^t Y_d(s-) dX_s + [X_d(t), Y_d(t)] \qquad (III-3) \\
&= \sum_{\alpha=1}^m \int_0^t (\nabla \xi_{s-})^{-1} [V, V_\alpha](\xi_{s-}) dZ_d^\alpha(s) \\
&\quad + \sum_{0<s\le t} (\nabla \xi_{s-})^{-1} \{ (\nabla \exp(\cdot)(\xi_{s-}))^{-1} V(\exp(\cdot)(\xi_{s-})) - V(\xi_{s-}) \\
&\quad - \sum_{k=1}^m \Delta Z_s^k [V, V_k](\xi_{s-}) \}.
\end{aligned}$$

Since,

$$\frac{d}{ds} (\nabla \exp(\cdot))^{-1} V(x) = -\sum_{k=1}^m \Delta Z_s^k (\nabla \exp(\cdot))^{-1} [V, V_k](x)$$

holds, we have

$$\begin{aligned}
&(\nabla \exp(\cdot))^{-1} V(\exp(\cdot)(x)) - V(x) \\
&= -\sum_{k=1}^m \Delta Z_s^k (\nabla \exp(\sum_{\alpha=1}^m \Delta Z_\theta^\alpha V_\alpha))^{-1} [V, V_k](x),
\end{aligned}$$

where $0 < \theta < 1$, by the mean value theorem. Substitute the above to (III-3). Then we get (III-2). $\square$

LEMMA III-3. (c.f.[3]) *Let $l \neq 0$ be a vector in $\mathbf{R}^d$. Suppose that*

$$l^* (\nabla \xi_t)^{-1} V(\xi_t) = 0, \quad for \quad t \in [0, \tau),$$

where τ is a stopping time such that $0 < \tau \leq T$ a.s. Then, for $t \in [0, \tau)$ a.s.,

$$l^*(\nabla \xi_t)^{-1}[V, V_k](\xi_t) = 0, \quad k = 1, 2, \cdots, m.$$

Proof. We consider the semi-martingale $Y_t = (\nabla \xi_{t\wedge\tau})^{-1}V(\xi_{t\wedge\tau})$. It has a unique Meyer decomposition $Y_t = M_t + A_t$, where M_t is a local martingale and A_t is a predictable process of bounded variation. If $l^*Y_t = 0$, then $l^*M_t = 0$ holds. Further let $M_c(t)$ and $M_d(t)$ be continuous and discontinuous local martingales, respectively, such that $M(t) = M_c(t) + M_d(t)$. Then $l^*M_t = 0$ implies $l^*M_c(t) = 0$ and $l^*M_d(t) = 0$. Consequently, we have by Lemma III-2, if $s < \tau$

$$\sum_{k=1}^{m} l^*(\nabla \xi_s)^{-1}(\nabla \exp(\cdot))^{-1}[V, V_k](\exp(\cdot)(\xi_s))z^k = 0, \quad a.e. \quad \nu,$$

where $\exp(\cdot) = \exp(\sum_{\alpha=1}^{m} \Delta Z^\alpha(\theta)V_\alpha)$. Define $m \times d$ matrix by $[V, \tilde{\mathbf{V}}](x) = ([V, V_1](x), \cdots, [V, V_m](x))$. Then we get, if $s < \tau$,

$$\int_{|z|<\rho} l^*(\nabla \xi_s)^{-1}(\nabla \exp(\cdot))^{-1}[V, \tilde{\mathbf{V}}](\exp(\cdot)(\xi_s))zz^*$$
$$[V, \tilde{\mathbf{V}}](\exp(\cdot)(\xi_s))^*(\nabla \exp(\cdot))^{-1,*}(\nabla \xi_s)^{-1,*}l\nu(dz) = 0.$$

Divide the above by $v(\rho)$ and let ρ tend to 0. Then we obtain

$$l^*(\nabla \xi_s)^{-1}[V, \tilde{\mathbf{V}}]B[V, \tilde{\mathbf{V}}]^*(\nabla \xi_s)^{-1,*}l = 0, \quad if \quad s < \tau,$$

where $B := \liminf_{\rho \to 0}(v(\rho))^{-1}V(\rho)$. Since B is non-degenerate, if $s < \tau$, we get; $l^*(\nabla \xi_s)^{-1}[V, \tilde{\mathbf{V}}] = 0$ or

$$l^*(\nabla \xi_s)^{-1}[V, V_k] = 0, \quad k = 1, 2, \cdots, m.$$

Next consider the bounded variation part A_t. It is equal to

$$-\sum_{k=1}^{m} \int_0^{t\wedge\tau} (\nabla \xi_s)^{-1}[V, V_k](\xi_s)ds,$$

since the terms of A_t involving $l^*(\nabla \xi_s)^{-1}[V, V_k], \quad k \geq 1$, are 0. Therefore, $l^*A_t = 0$ implies

$$\sum_{k=1}^{m} l^*(\nabla \xi_s)^{-1}[V, V_k] = 0, \quad \text{if} \quad s < \tau.$$

Thus, we get the result. □

Proof of Proposition III-1. If (II-11) is given, because $\nabla\xi_t(x)$ is invertible by Lemma III-1, we get for any non-zero vector $l(\neq 0) \in \mathbf{R}^d$;

$$< l, \quad (\nabla\xi_t(x))^{-1}[V, V_k](\xi_t(x)) > \neq 0$$

for any given vector field V and $k = 1, 2, ..., m$. Then, by Lemma III-3,

$$< l, \quad (\nabla\xi_t(x))^{-1}V_k(\xi_t(x)) > \neq 0.$$

Thus, because of $\nabla\xi_{r,t} = \nabla\xi_t(\nabla\xi_r)^{-1}$, we get

$$< l, \quad \nabla\xi_{r,t}(\xi_r)V_k(\xi_r) > \neq 0.$$

Thus, by Lemma III-1, we get

$$\begin{aligned} \det(\Psi_r\Psi_r^*)^{-1} &= det((\nabla\xi_{r,t}(\xi_r))^*\tilde{\mathbf{V}}(\xi_r)(\tilde{\mathbf{V}}(\xi_r))^{*,-1}(\nabla\xi_{r,t}(\xi_r))^{-1}) \\ &\neq 0. \end{aligned}$$

Thus we get (II-1) in Proposition II-1;

$$\|(\det \int_0^t \Psi_r\Psi_r^* dr)^{-1}\|_p < \infty. \quad \Box$$

From the Sobolev inequality;

$$\sup |H(x)| \leq C \sum_{|k|=d+1} \int |H^{(k)}(x)|dx$$

for smooth functions H with compact support in $\mathbf{R}^d$, we deduce that;

$$\sup_{|x|\leq\rho} |H(x)| \leq C \sum_{|k|\leq d+1} \int_{\{|x|\leq\rho+1\}} |H^{(k)}(x)|dx \qquad (III-4)$$

for a C which does not depend on ρ (c.f. [5]).

Lemma III-4. (see [5]) *Let $H_1(w, x_1, u)$ and $H_2(w, x_1, x, u), x_1 \in \mathbf{R}^d, x \in \mathbf{R}^d, u \in E$(a parameter space), be random functions such that*

$$\begin{aligned} \|\sup_u |H_1(x_1, u)|\|_p &\leq Q_p(x_1), \\ \|\sup_u |H_2^{(k)}(x_1, x, u)|\|_p &\leq Q_{k,p}(x), \qquad (III-5) \end{aligned}$$

for $p \geq 1, k \in \mathbf{N}^d$, some functions $Q_p, Q_{k,p}$ with at most polynomial growth, and $H_2^{(k)}$ are the derivatives with respect to x. Then the function

$$H : (x_1, u) \mapsto H_2(x_1, H_1(x_1, u), u)$$

satisfies an estimate similar to the one for H_1; for any p, there exists a function $\bar{Q}_p$ with at most polynomial growth such that

$$\| \sup_u |H(x_1, u)| \|_p \leq \bar{Q}_p(x_1). \qquad (III-6)$$

LEMMA III-5. (See [4] and [5]) *We have*

$$\| \sup_{0 \leq s \leq t \leq T} |D^{(k)} \xi_{s,t}(x)| \|_p \leq Q_{k,p}(x) \qquad (III-7)$$

for some functions $Q_{k,p}$ with at most polynomial growth, and where the supremum is relative to the couples (s, r).

More generally, we can get also that, for any stopping time σ, the process $\xi_{\sigma,t}(x)$ is the solution of (II-7) with initial value x at time σ, and

$$\begin{aligned} \| \sup_{\sigma \leq t \leq T} |\xi_{\sigma,t}(x)| \|_p &\leq C_p(1 + |x|), \\ \| \sup_{\sigma \leq t \leq T} |D^{(k)} \xi_{\sigma,t}(x)| \|_p &\leq C_p, \qquad (III-8) \end{aligned}$$

for $k \neq 0$ and where C_p does not depend on σ.

Proof of the Theorem. Consider a function

$$\phi(\rho, z, x) := x + \rho |z|^{-1} (\exp(\sum_{\alpha=1}^m z^\alpha V_\alpha)(x) - x), \rho \geq 0, |z| \leq 1,$$

and the random map

$$\begin{aligned} & R(x_0, \rho_1, t_1, z_1, \cdots, \rho_k, t_k, z_k) \\ := \; & \xi_{t_k,t} \circ \phi(\rho_k, z_k, \cdot) \circ \xi_{t_{k-1},t_k} \circ \cdots \circ \xi_{t_1,t_2} \circ \phi(\rho_1, z_1, \cdot) \circ \xi_{0,t_1}(x_0) \end{aligned}$$

for $0 \leq t_1 < t_2 < \cdots < t_k \leq t$. Then, for $\tau = ((t_1, z_1), \cdots, (t_k, z_k))$, we get

$$\begin{aligned} \xi_t \circ \varepsilon_\tau^+ &= \xi_t \circ \varepsilon_{t_1,z_1}^+ \circ \varepsilon_{t_2,z_2}^+ \circ \cdots \circ \varepsilon_{t_k,z_k}^+ \\ &= R(x_0, |z_1|, t_1, z_1, \cdots, |z_k|, t_k, z_k), \end{aligned}$$

and

$$\begin{aligned} D_\tau F &:= D_\tau \xi_t \\ &= \xi_t \circ \varepsilon_\tau^+ - \xi_t \\ &= R(x_0, |z_1|, t_1, z_1, \cdots, |z_k|, t_k, z_k) - \xi_t \\ &= \int_0^{|z_1|} \cdots \int_0^{|z_k|} \frac{\partial^k}{\partial \rho_1 \cdots \partial \rho_k} R(x_0, \rho_1, \cdots, t_k, z_k) d\rho_k \cdots d\rho_1. \end{aligned}$$

(1). In order to get the boundedness of (II-4) in (1) of Proposition II-1, we can use

$$\operatorname{ess\,sup}_{\tau}\{\frac{|D_\tau F|}{\prod |z_j|}\} \le \sup\{|\frac{\partial^k}{\partial\rho_1\cdots\partial\rho_k}R(x_0,\rho_1,\cdots,t_k,z_k)|;$$
$$0\le\rho_j\le 1, 0\le t_1<t_2<\cdots<t_k\le t, |z_j|\le 1\}.$$

To estimate the supremum of

$$\frac{\partial^k}{\partial\rho_1\cdots\partial\rho_k}R(x_0,\rho_1,t_1,z_1,\cdots,\rho_k,t_k,z_k),$$

we use that the derivatives of $\phi(\rho,z,x)$ with respect to ρ;

$$\frac{\partial}{\partial\rho}\phi(\rho,z,x)=|z|^{-1}(\exp(\sum_{\alpha=1}^m z^\alpha V_\alpha)(x)-x), \rho\ge 0, |z|\le 1,$$

is bounded. Further, from Lemma II-1, the derivatives of $\phi(\rho,z,x)$ with respect to x are bounded. Thus, it is reduced to estimate the moments of variables of type;

$$D^{(k')}\xi_{t_j,t_{j+1}}\circ\phi(\rho_j,z_j,\cdot)\circ\xi_{t_{j-1},t_j}\circ\cdots\circ\phi(\rho_1,z_1,\cdot)\circ\xi_{0,t_1}(x_0).$$

From Lemma III-5, we can estimate of $D^{(k')}\xi_{t_j,t_{j+1}}(x)$ and $\xi_{t_l,t_{l+1}}(x)$ for all j and $l\le k$. Thus we obtain the (II-4) of Proposition II-1.

(2). To get the Condition (2) of Proposition II-1, we use the fact;

$$\begin{aligned}\xi_t\circ\varepsilon^+_{r,z} &= \xi_{r,t}(\xi_r+\sum_{\alpha=1}^m z^\alpha V_\alpha(\xi_r)+c(\xi_r,z))\\ &= \xi_{r,t}(\exp(\sum_{\alpha=1}^m z^\alpha V_\alpha)(\xi_r))\end{aligned}$$

is differentiable with respect to z at $z=0$. Thus from the fact, we get

$$\begin{aligned}D_{r,z}\xi_t &= \xi_t\circ\varepsilon^+_{r,z}-\xi_t\\ &= \xi_{r,t}(\exp(\sum_{\alpha=1}^m z^\alpha V_\alpha)(\xi_r))-\xi_t\\ &= \xi_{r,t}\circ\phi(|z|,z,\cdot)\circ\xi_r(x_0)-\xi_{r,t}\circ\phi(0,z,\cdot)\circ\xi_r(x_0)\\ &= R(x_0,|z|,r,z)-R(x_0,0,r,z)\\ &= \int_0^{|z|}\int_0^{\bar\rho}\frac{\partial^2}{\partial\bar\rho\partial\rho}R(x_0,\bar\rho,r,z,\rho,t,|z|)d\rho d\bar\rho.\end{aligned}$$

Therefore, from the fact;

$$R(x_0,0,t,z) = \xi_{r,t} \circ \phi(0,z,\cdot) \circ \xi_r(x_0),$$

and

$$\begin{aligned}\frac{\partial}{\partial \rho} R(x_0,0,t,z) &= \nabla \xi_{r,t}(\xi_r(x_0)) \frac{\partial}{\partial \rho}\phi(\rho,z,\xi_r(x_0))|_{\rho=0} \\ &= \nabla \xi_{r,t}(\xi_r)\frac{1}{|z|}\nabla(\exp(\sum_{\alpha=1}^m z^\alpha V_\alpha)(\xi_r)-\xi_r),\end{aligned}$$

we get that

$$|z|\frac{\partial}{\partial \rho}R(x_0,0,t,z) = \nabla\xi_{r,t}(\xi_r)\nabla(c(\xi_r,z)+\sum_{\alpha=1}^m z^\alpha V_\alpha(\xi_r)).$$

Thus we get

$$\begin{aligned}\Psi_r z &= \nabla \xi_{r,t}(\xi_r)\tilde{\mathbf{V}}(\xi_r)z \\ &= \sum_{i=1}^d \frac{\partial}{\partial x^i}\xi_{r,t}(\xi_r(x))\tilde{\mathbf{V}}^i(\xi_r(x))z \\ &= |z|\frac{\partial}{\partial\rho}R(x_0,0,t,z) - \nabla\xi_{r,t}(\xi_r)\nabla c(\xi_r,z)\end{aligned}$$

Therefore, we get

$$\begin{aligned}&D_{r,z}\xi_t - \Psi_r z \\ &= \int_0^{|z|}\int_0^{\bar\rho}\frac{\partial^2}{\partial\bar\rho\partial\rho}R(x_0,\bar\rho,r,z,\rho,t,|z|)d\rho d\bar\rho - |z|\frac{\partial}{\partial\rho}R(x_0,0,t,z) \\ &\qquad + \nabla\xi_{r,t}(\xi_r)\nabla c(\xi_r,z) \\ &= \sum_{i,j}^d\int_0^{|z|}\int_0^{\bar\rho}\frac{\partial^2}{\partial x^i\partial x^j}R(x_0,\rho,r,z)\frac{\partial}{\partial\rho}\phi^i(\rho,z,\xi_r)\frac{\partial}{\partial\rho}\phi^j(\rho,z,\xi_r)d\rho d\bar\rho \\ &\qquad + \nabla\xi_{r,t}(\xi_r)\nabla c(\xi_r,z) \\ &= \sum_{i,j}^d\int_0^{|z|}\int_0^{\bar\rho}\frac{\partial^2}{\partial x^i\partial x^j}(\xi_{r,t}\circ\phi(\rho,z,\xi_r))\frac{\partial}{\partial\rho}\phi^i(\rho,z,\xi_r)\frac{\partial}{\partial\rho}\phi^j(\rho,z,\xi_r)d\rho d\bar\rho \\ &\qquad + \nabla\xi_{r,t}(\xi_r)\nabla c(\xi_r,z).\end{aligned}$$

The moments of the first and second derivatives are proved to be bounded from (III-8). Also, the variables $\exp(\sum_{\alpha=1}^m z^\alpha V_\alpha)(\xi_r) - \xi_r$ and $c(\xi_r,z)$

are, respectively, of order $|z|^2$ and $|z|^r$ because V_α are bounded functions (c.f.[3] and [4]). Therefore, this expression is order $|z|^{r \wedge 2}$ for $|z| < 1$. Thus, we can get the (II-5) of (2) in Proposition II-1. Thus, from the Proposition II-1, we get the result. $\square$

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