Kangweon-Kyungki Math. Jour. **8** (2000), No. 1, pp. 19–26

SOME CONDITIONS FOR COMAXIMALITY OF IDEALS

Sung Hun Ahn

Abstract. In this paper, it is shown that if R is a commutative ring with identity and there exists a multiplicatively closed subset S of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$ and $I_1 R_S, I_2 R_S, \cdots, I_n R_S$ are pairwise comaximal, then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \cap_{i=1}^{n} (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$.

1. Introduction

Throughout this paper, R will be a commutative ring with identity and I, J and $I_1, I_2, \cdots, I_n$ are ideals of R, unless otherwise stated. If S is a multiplicatively closed subset of R, then R_S is the quotient ring of R with respect to S and IR_S is the extension of I in R_S. $Ass_R(R/I)$ denotes the set of all associated prime ideals of I. We denote by $(I :_R J) = \{ r \in R : rJ \subseteq I \}$ the annihilator of $(J + I)/I$. It is well known that if $I_1, I_2, \cdots, I_n$ are pairwise comaximal ideals of R, then $I_1 \cap I_2 \cap \cdots \cap I_n = I_1 I_2 \cdots I_n$. In[3], Ratliff gave some variations of this. He showed that if $I_1, I_2, \cdots, I_n$ are ideals of a Noetherian ring R, then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^{n} (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$ if and only if each $P \in Ass_R(R/I_1 I_2 \cdots I_n)$ contains exactly one of the ideals $I_1, I_2, \cdots, I_n$. The purpose of this paper is to generalize this result for ideals in a commutative ring with identity.

In section 2, it is shown that if R is a commutative ring with identity and there exists a multiplicatively closed subset S of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$ and $I_1 R_S, I_2 R_S, \cdots, I_n R_S$ are pairwise comaximal, then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^{n} (I_i :_R$

Received January 5, 2000.
1991 Mathematics Subject Classification: 13A15, 13A30, 13C12.
Key words and phrases: Associated prime ideals, Comaximal ideals.
This paper was supported by Dongguk University Research Fund, 1999

$I_1 \cdots I_{i-1}I_{i+1} \cdots I_n)$. It is also shown that if $I_1 I_2 \cdots I_n$ has a finite primary decomposition and each I_i is finitely generated, then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^{n} (I_i :_R I_1 \cdots I_{i-1}I_{i+1} \cdots I_n)$ if and only if there exists a multiplicatively closed subset S of R such that $I_1 R_S, I_2 R_S, \cdots, I_n R_S$ are pairwise comaximal and

$$S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset.$$

Finally, we show that if $I_1, I_2, \cdots, I_n$ are ideals in a Noetherian ring R, then each $P \in Ass_R(R/I_1 I_2 \cdots I_n)$ contains exactly one of the ideals $I_1, I_2, \cdots, I_n$ if and only if $(R/(I_1 I_2 \cdots I_n))_S \cong \prod_{j=1}^{n} (R/I_j)_{S_j}$, where

$$S = R \setminus \bigcup \{P : P \in Ass_R(R/I_1 I_2 \cdots I_n)\}$$

and

$$S_j = R \setminus \bigcup \{P : P \in Ass_R(R/I_1 I_2 \cdots I_n) \text{ and } I_j \in P\}.$$

2. Main Results

We begin this section by listing some of known results that will be used in this paper.

LEMMA 2.1. *Let R be a commutative ring with identity, $I_1, I_2, \cdots, I_n, I, J$ and K ideals of R and S a multiplicatively closed subset of R. Then*

(1) $(IJ)R_S = IR_S JR_S$.
(2) $(I \cap J)R_S = IR_S \cap JR_S$.
(3) $(I :_R J)R_S \subseteq (IR_S :_{R_S} JR_S)$.
If J is finitely generated then the equality holds.
(4) $I \subseteq (IR_S) \cap R$.
The equality holds if and only if $S \cap Z(R/I) = \emptyset$, where $Z(R/I) = \{ x \in R : (I :_R x) \neq I \}$.
(5) *If I has a finite primary decomposition then* $Z(R/I) = \cup \{ P : P \in Ass_R(R/I) \}$.

(6) $(I_1 \cap I_2 \cap \cdots \cap I_n \;:_R\; J) = \bigcap_{i=1}^{n} (I_i \;:_R\; J)$.

(7) $(J \;:_R\; I_1 + \cdots + I_n)) = \bigcap_{i=1}^{n} (J \;:_R\; I_i)$.

(8) *If Q is a primary component of I then $I \subset (I \;:_R\; Q)$.*

(9) $(I \;:_R\; JK)) = ((I \;:_R\; J) \;:_R\; K)$.

Proof. This is straightforward. □

The following results concerning pairwise comaximal ideals are well known.

Lemma 2.2. *Let R be a commutative ring with identity and let $I_1, I_2, \cdots, I_n$ be pairwise comaximal ideals of R. Then*

(1) *For each i, I_i and $(I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)^m$ are comaximal, for all positive integer m.*

(2) *For each i, $(I_i \;:_R\; (I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)^m) = I_i$, for all positive integer m.*

(3) $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^{n} (I_i \;:_R\; (I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)^m)$, *for all positive integer m.*

Proof. It is easy to see that if I and J are comaximal then $(I \;:_R\; J) = I$. Hence (2) follows from (1). To prove (1), it suffices to show that if I, J and K are pairwise comaximal ideals then I and JK are comaximal. Since $R = (I + J)(I + K) = I^2 + IJ + IK + JK \subseteq I + JK$, hence I and JK are comaximal. For (3), the last equality follows from (1). To show the first equality, it may be assumed that $n = 2$. Let $x \in I_1 \cap I_2$. Since I_1 and I_2 are comaximal, there exist $a \in I_1$, $b \in I_2$ such that $a + b = 1$. $x = ax + bx \in I_1 I_2$. Thus $I_1 \cap I_2 \subseteq I_1 I_2$. The opposite inclusion is clear. Hence $I_1 I_2 = I_1 \cap I_2$. □

The following is the Chinese remainder theorem which is well known, so we omit the proof.

Theorem 2.3. *Let R be a commutative ring with identity and let $I_1,\ I_2, \cdots, I_n$ be pairwise comaximal ideals of R. If $b_1,\ b_2,\ \cdots,\ b_n$ are elements of R, then there exists $b \in R$ such that $b \equiv b_i \pmod{I_i}$, for $i = 1,\ 2,\ \cdots,\ n$.*

LEMMA 2.4. *Let R be a commutative ring with identity and let $I_1,\ I_2, \cdots, I_n$ be pairwise comaximal ideals of R. Then A map $f\ :\ R \longrightarrow \prod_{j=1}^{n} R/I_j$ defined by $f(r) = (r+I_1,\ r+I_2,\ \cdots,\ r+I_n)$ is an epimorphism of rings if and only if $I_1, I_2, \cdots, I_n$ are pairwise comaximal. In this case, $R/(I_1 I_2 \cdots I_n) \cong \prod_{j=1}^{n} R/I_j$.*

Proof. Let the map $f\ :\ R \longrightarrow \prod_{j=1}^{n} R/I_j$ defined by $f(r) = (r + I_1,\ r + I_2,\ \cdots,\ r + I_n)$ be an epimorphism of rings. For any $i \in \{1,\ 2,\ \cdots,\ n\}$, by resubscripting, if necessary, we may assume that $i = 1$. Since f is onto, there exists $r \in R$ such that $f(r) = (1 + I_1,\ I_2,\ \cdots,\ I_n)$. Thus $r - 1 \in I_1$ and $r \in I_2 \cap I_3 \cap \cdots \cap I_n$. This implies that there exists $a \in I_1$ such that $r - 1 = a$. Thus $1 = r - a \in I_1 + I_j$, for $j = 2,\ 3,\ \cdots,\ n$. Hence $I_1, I_2, \cdots, I_n$ are pairwise comaximal. The converse follows from the Chinese remainder theorem. □

Theorem(2.5) is one of our main theorems in this paper.

THEOREM 2.5. *Let R be a commutative ring with identity and let $I_1,\ I_2, \cdots, I_n$ be ideals of R. If there exists a multiplicatively closed subset S of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$ and $I_1 R_S, I_2 R_S, \cdots, I_n R_S$ are pairwise comaximal, then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^{n} (I_i\ :_R\ I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$.*

Proof. Let S be a multiplicatively closed subset of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$ and $I_1 R_S, I_2 R_S, \cdots, I_n R_S$ are pairwise comaximal. By lemma(2.1)(4), $(I_1 I_2 \cdots I_n) R_S \cap R = I_1 I_2 \cdots I_n$. Since $I_1 R_S, I_2 R_S, \cdots, I_n R_S$ are pairwise comaximal,

$$I_1 R_S I_2 R_S \cdots I_n R_S = I_1 R_S \cap I_2 R_S \cap \cdots \cap I_n R_S$$

$$= \bigcap_{i=1}^{n} (I_i R_S\ :_{R_S}\ I_1 R_S \cdots I_{i-1} R_S I_{i+1} R_S \cdots I_n R_S).$$

By contracting these ideals to R, we get

$$I_1 I_2 \cdots I_n \subseteq I_1 \cap I_2 \cap \cdots \cap I_n \subseteq (I_1 R_S \cap I_2 R_S \cap \cdots \cap I_n R_S) \cap R$$

$$= (I_1R_SI_2R_S\cdots I_nR_S)\cap R = (I_1I_2\cdots I_n)R_S\cap R = I_1I_2\cdots I_n$$

and

$$I_1I_2\cdots I_n \subseteq I_1\cap I_2\cap\cdots\cap I_n$$

$$\subseteq \bigcap_{i=1}^{n}(I_i :_R I_1\cdots I_{i-1}I_{i+1}\cdots I_n)$$

$$\subseteq ((\bigcap_{i=1}^{n}(I_i :_R I_1\cdots I_{i-1}I_{i+1}\cdots I_n))R_S)\cap R$$

$$= (\bigcap_{i=1}^{n}((I_i :_R I_1\cdots I_{i-1}I_{i+1}\cdots I_n)R_S))\cap R$$

$$\subseteq (\bigcap_{i=1}^{n}(I_iR_S :_{R_S} I_1R_S\cdots I_{i-1}R_SI_{i+1}R_S\cdots I_nR_S))\cap R$$

$$= (I_1R_S\cap I_2R_S\cap\cdots\cap I_nR_S)\cap R = I_1I_2\cdots I_n.$$

Thus $I_1I_2\cdots I_n = I_1\cap I_2\cap\cdots\cap I_n = \bigcap_{i=1}^{n}(I_i :_R I_1\cdots I_{i-1}I_{i+1}\cdots I_n)$.□

As an application of the result in [3], Ratliff proved the following theorem. This gives an example of ideals for which the conditions in theorem(2.5) are satisfied. We refer the reader to [3, theorem(2.7)] for a proof.

THEOREM 2.6. *Let R be a Noetherian ring and $a_1,\ a_2,\cdots$ and a_n nonunit elements on R. Let each permutation of $a_1,\ a_2,\cdots$ and a_n be an R-sequence. If $I_1,\ I_2,\cdots$ and I_m are ideals generated by disjoint subset of $a_1,\ a_2,\cdots$ and a_n, then $I_1^{k_1}I_2^{k_2}\cdots I_m^{k_m} = I_1^{k_1}\cap I_2^{k_2}\cap\cdots I_m^{k_m} = \bigcap_{j=1}^{m}(I_j^{k_j} :_R (I_1\cdots I_{j-1}I_{j+1}\cdots I_m)^t)$, for all positive integer $k_1,\cdots,\ k_m$ and t.*

Theorem(2.7) is another main theorem in this paper.

Theorem 2.7. *Let R be a commutative ring with identity and let $I_1,\ I_2, \cdots, I_n$ be finitely generated ideals in R. Let $I_1 I_2 \cdots I_n$ have a finite primary decomposition. Then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^{n} (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$ if and only if there exists a multiplicatively closed subset S of R such that $I_1 R_S, I_2 R_S, \cdots, I_n R_S$ are pairwise comaximal and $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$.*

Proof. Assume that

$$I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n$$

$$= \bigcap_{i=1}^{n} (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n).$$

Let $S = R \setminus Z(R/(I_1 I_2 \cdots I_n))$. Then S is a multiplicatively closed subset of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$. Since $I_1 I_2 \cdots I_n$ has a finite primary decomposition, it follows from lemma(2.4)(5) that $Z(R/(I_1 I_2 \cdots I_n)) = \cup \{ P : Ass_R(R/(I_1 I_2 \cdots I_n)) \}$. Thus, to show that $I_1 R_S, I_2 R_S, \cdots, I_n R_S$ are pairwise comaximal, it suffices to show that no $P \in Ass_R(R/I_1 I_2 \cdots I_n)$ contains more than one of the I_j. For any $i \neq k \in \{1,\ 2,\ \cdots,\ n\}$, by resubscripting, if necessary, we may assume that $i = 1,\ k = 2$. Then we have

$$I_1 \cap I_2 \cap \cdots \cap I_n \subseteq (I_1 \cap I_2 \cap \cdots \cap I_n :_R I_1 + I_2) = \bigcap_{i=1}^{n} (I_i :_R I_1 + I_2)$$

$$= (I_1 :_R I_2) \cap (I_2 :_R I_1) \cap (I_3 :_R I_1 + I_2) \cap \cdots \cap (I_n :_R I_1 + I_2)$$

$$\subseteq (I_1 :_R I_2) \cap (I_2 :_R I_1) \cap (I_3 :_R I_1 I_2) \cap \cdots \cap (I_n :_R I_1 I_2)$$

$$\subseteq \bigcap_{i=1}^{n} (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$$

$$= I_1 \cap I_2 \cap \cdots \cap I_n.$$

This show that, for all $i \neq k \in \{1,\ 2,\ \cdots,\ n\}$,

$$I_1 \cap I_2 \cap \cdots \cap I_n = (I_1 \cap I_2 \cap \cdots \cap I_n :_R I_i + I_k).$$

By hypothesis, for all $i \neq k \in \{1,\ 2,\ \cdots,\ n\}$,

$$I_1 I_2 \cdots I_n = (I_1 I_2 \cdots I_n \ :_R\ I_i + I_k).$$

Hence $I_1 I_2 \cdots I_n = (I_1 I_2 \cdots I_n \ :_R\ (I_i + I_k)^m)$, for all $m \geq 1$ and for all $i \neq k \in \{1,\ 2,\ \cdots,\ n\}$. If $P \in Ass_R(R/I_1 I_2 \cdots I_n)$ such that $I_i + I_k \subseteq P$, then $(I_i + I_k)^m$ is contained in a $P-$primary component Q of $I_1 I_2 \cdots I_n$, for large m. This implies that

$$I_1 I_2 \cdots I_n \subset (I_1 I_2 \cdots I_n \ :_R\ Q)$$

$$\subseteq (I_1 I_2 \cdots I_n \ :_R\ (I_i + I_k)^m) = I_1 I_2 \cdots I_n,$$

which is a contradiction. Hence $I_1 R_S, I_2 R_S, \cdots, I_n R_S$ are pairwise comaximal. The converse follows from theorem(2.7). □

If R is a Noetherian ring then each ideal has a finite primary decomposition. Hence we have the following corollary.

Corollary 2.8. *Let R be a Noetherian ring and let $I_1,\ I_2, \cdots, I_n$ be ideals in R. Then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^{n} (I_i \ :_R\ I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$ if and only if there exists a multiplicatively closed subset S of R such that $I_1 R_S, I_2 R_S, \cdots, I_n R_S$ are pairwise comaximal and $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$.*

Proof. This immediately follows from theorem(2.7). □

The following theorem is proved by Ratliff in [3].

Theorem 2.9. *Let R be a Noetherian ring and let $I_1,\ I_2, \cdots, I_n$ be ideals in R. Then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^{n} (I_i \ :_R\ I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$ if and only if each $P \in Ass_R(R/I_1 I_2 \cdots I_n)$ contains exactly one of the ideals $I_1, I_2, \cdots, I_n$*

Corollary 2.10. *Let R be a Noetherian ring and let $I_1,\ I_2, \cdots, I_n$ be ideals in R. Then there exists a multiplicatively closed subset S of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$ and $I_1 R_S, I_2 R_S, \cdots, I_n R_S$ are pairwise comaximal if and only if each $P \in Ass_R(R/I_1 I_2 \cdots I_n)$ contains exactly one of the ideals $I_1, I_2, \cdots, I_n$*

Proof. This follows from corollary(2.8) and theorem(2.9). □

The following corollary is analogous to lemma(2.4).

COROLLARY 2.11. *Let R be a Noetherian ring and let $I_1,\ I_2, \cdots, I_n$ be ideals in R. Then each $P \in Ass_R(R/I_1I_2\cdots I_n)$ contains exactly one of the ideals $I_1, I_2, \cdots, I_n$ if and only if $(R/(I_1I_2\cdots I_n))_S \cong \prod_{j=1}^{n}(R/I_j)_{S_j}$, where*

$$S = R\setminus \bigcup\{P\ :\ P\in Ass_R(R/I_1I_2\cdots I_n)\}$$

and

$$S_j = R\setminus \bigcup\{P\ :\ P\in Ass_R(R/I_1I_2\cdots I_n) \text{ and } I_j\in P\}.$$

Proof. By corollary(2.10), each $P \in Ass_R(R/I_1I_2\cdots I_n)$ contains exactly one of the ideals $I_1, I_2, \cdots, I_n$ if and only if $I_1R_S, I_2R_S, \cdots, I_nR_S$ are pairwise comaximal. Thus, by lemma(2.4),

$$R_S/((I_1I_2\cdots I_n)R_S) \cong \prod_{j=1}^{n} R_S/(I_jR_S)$$

It is easy to see that $R_S/(I_jR_S) \cong R_{S_j}/(I_jR_{S_j})$ and $(R/I)_T \cong R_T/(IR_T)$ for all ideals I and for all multiplicatively closed subset T of R. This completes the proof. □

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Department of Mathematics
Dongguk University
Seoul 100–715, Korea