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CHARACTERIZATIONS OF INFINITELY DIVISIBLE AND TYPE G PROCESSES

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Abstract. We generalize LePage-type representation of infinitely divisible processes and survey type G processes. Finally, we get integral representation and some inequality of processes of type G.

1. Introduction

Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of i.i.d. random vectors related to some measure and let Γ_j be the j-th arrival time of unit rate Poisson process. Consider the series representation

$$\sum_{j=1}^{\infty} R(\Gamma_j, \xi_j) f(\xi_j),$$

where, R is a function related to control measure and Borel measurable function and f is a function belonging to Musielak-Orlicz space. Series representations involving arrival times in a Poisson process have been given by Ferguson and Klass [1], for real independent increment processes without Gaussian components and with positive jumps. LePage series representation is developed for infinitely divisible processes by Rajput and Rosinski ([3],[4],[5],[7]) and tail behavior of subadditive functionals of paths of infinitely divisible processes is obtained ([6]).

Chapter 2 is to review some basic definitions and some example. In chapter 3, we generalize LePage-type representation of infinitely divisible processes. In chapter 4, we survey a large class of infinitely divisible processes, known as, type G processes and get integral representation and some inequality of processes of type G.

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2. Preliminaries

We denote, by S, an arbitrary non-empty set and, by $\mathcal{S}$, a δ-ring of subsets of S. Let $\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}$ be a real stochastic process defined on some probability space $(\Omega, \mathcal{F}, P)$. We begin with some definitions.

DEFINITION 2.1. *A random measure Λ is said to be an independently scattered random measure if for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{S}$, the random variables $\Lambda(A_n)$, $n = 1, 2, \cdots$ are independent and if $\cup_n A_n$ belong to $\mathcal{S}$ then we also have $\Lambda(\cup_n A_n) = \sum_n \Lambda(A_n)$ a.s, where the series is assumed to converge almost surely.*

DEFINITION 2.2. *A random measure Λ is said to be an independently scattered infinitely divisible random measure if for each $A \in \mathcal{S}$, $\Lambda(A)$ is infinitely divisible random variable and Λ is independently scattered random measure.*

We know that characteristic function of infinitely divisible random variable $\Lambda(A)$ can be written in the Khintchin-Lévy form;

$$Ee^{iu\Lambda(A)}=\exp\{iu\nu_0(A)-\frac{1}{2}u^2\nu_1(A)+\int_R e^{iux}- 1 - iuxI_{|x|\le\tau}(x)F_A(dx)\} \quad (2.1)$$

for some $\tau > 0$, where $-\infty < \nu_0(A) < \infty, 0 \le \nu_1(A) < \infty$ and F_A is a Lévy measure on R.

DEFINITION 2.3. *A σ-finite measure ν on $\sigma(\mathcal{S})$ is said to be a control measure of the random measure Λ if Λ and ν have the same families of zero sets.*

EXAMPLE 2.1. *Let ν_0, ν_1 and $F_.$ be as in (2.1) and define*

$$\nu(A) := |\nu_0|(A) + \nu_1(A) + \int_R \min\{1, x^2\}F_A(dx), \quad A \in \mathcal{S},$$

Then $\nu : \sigma(\mathcal{S}) \to [0, \infty]$ is a control measure of Λ, where $\sigma(\mathcal{S})$ is the smallest σ-field generated by $\mathcal{S}$.

¿From [2], we adopt the following definition of a process of type G.

DEFINITION 2.4. *A process $\{X_t, t \in T\}$ is of type G if there is a function $\psi : [0, \infty) \to R$ with completely monotone derivative on $(0, \infty)$,*

$\psi(0)=0$, and a σ-finite measure λ on the cylindrical σ-field of R^T such that

$$E\exp\sum_{j=1}^{n}\theta_j X_{t_j}=\exp\{-\int_{R^T}\psi(2^{-1}|\sum_{j=1}^{n}\theta_j s(t_j)|^2)\lambda(ds)\}. \qquad (2.2)$$

3. Series representation

Let λ be an arbitrary but fixed control measure of Λ.

LEMMA 3.1. *Let $F_{\cdot}$ be as in (2.1). Then there exists a unique σ-finite measure F on $\sigma(\mathcal{S})\times\mathcal{B}(R)$ such that*

$$F(A\times B)=F_A(B), \quad \textit{for all} \;\; A\in\mathcal{S}, B\in\mathcal{B}(R). \qquad (3.1)$$

Moreover, there exists a function $Q: S\times\mathcal{B}(R)\to[0,\infty]$ such that

(i) $Q(s,\cdot)$ is a Borel measure on $\mathcal{B}(R)$ for every $s\in\mathcal{S}$,

(ii) $Q(\cdot,B)$ is a Borel measurable function, for every $B\in\mathcal{B}(R)$,

(iii) $\int_{S\times R}h(s,x)F(ds,dx)=\int_S\left[\int_R h(s,x)Q(s,dx)\right]\lambda(ds)$ for every $\sigma(\mathcal{S})\times\mathcal{B}(R)$-measurable function $h: S\times R\to[0,\infty]$.

(iv) $\int_R(1\wedge x^2)Q(s,dx)<\infty$, for every $s\in S$,

(v) $\Phi_{\Lambda(A)}(u)=\exp\{\int_A K(u,s)\lambda(ds)\}$, where,

$$K(u,s)=iua(s)-\frac{1}{2}u^2\sigma^2(s)+\int_R(e^{iux}-1-iuxI_{\{|x|\le\tau\}}(x))Q(s,dx),$$

$$a(s)=\frac{d\nu_0}{d\lambda}(s), \quad \sigma^2(s)=\frac{d\nu_1}{d\lambda}(s).$$

(vi) $\lambda\{s\in\mathcal{S}: a(s)=\sigma^2(s)=Q(s,R)=0\}=0$.

Proof. By [3, Lemma 2.3], there exists a unique σ-finite measure F on $\sigma(\mathcal{S})\times\mathcal{B}(R)$ satisfying (3.1) and we can find a function $\rho: S\times\mathcal{B}(R)\to[0,\infty]$ such that $\rho(s,\cdot)$ is a Lévy measure on $\mathcal{B}(R)$ for every $s\in S$, $\rho(\cdot,B)$ is a Borel measurable function for every $B\in\mathcal{B}(R)$ and

$$\int_{S\times R}h(s,x)F(ds,dx)=\int_S\int_R h(s,x)\rho(s,dx)\lambda(ds)$$

for every $\sigma(\mathcal{S})\times\mathcal{B}(R)$-measurable function $h: S\times R\to[0,\infty]$.

Since λ and ν are equivalent σ-finite measures on $\sigma(\mathcal{S})$, there exists a strictly positive and finite version ψ of the Radon-Nikodym derivative $d\nu/d\lambda$. Put

$$Q(s,dx) = \psi(s)\rho(s,dx).$$

Then (i), (ii) and (iii) and (v) follow because

$$F_A(B) = F(A \times B) = \int_A \int_R I_B(x) Q(s,dx)\lambda(ds).$$

Since $\rho(s,\cdot)$ is a Lévy measure, (iv) is satisfied. Finally, note that $A_0 = \{s : a(s) = \sigma^2(s) = Q(s,R) = 0\}$ is a Λ-zero set, so that $\lambda(A_0) = 0$. $\square$

Let q be a non-negative number such that

$$E|\Lambda(A)|^q < \infty \quad \text{for all } A \in \mathcal{S}.$$

Define

$$U(u,s) = ua(s) + \int_R (uxI_{\{|ux|\le 1\}}(x) - uxI_{\{|x|\le 1\}}(x))Q(s,dx)$$

and for $0 \le p \le q$, $u \in R$ and $s \in \mathcal{S}$,

$$\Phi_p(u,s) = U^*(u,s) + u^2\sigma^2(s) + V_p(u,s),$$

where

$$U^*(u,s) = \sup_{|c|\le 1} |U(cu,s)|,$$

$$V_p(u,s) = \int_{-\infty}^{\infty} \{|ux|^p I_{\{|ux|>1\}} + |ux|^2 I_{\{|ux|\le 1\}}\} Q(s,dx).$$

We define the so-called Musielak-Orlicz space

$$L_{\Phi_p}(\mathcal{S},\lambda) = \{f \in L_0(\mathcal{S},\lambda) : \int_S \Phi_p(|f(s)|,s)\lambda(ds) < \infty\}.$$

The space $L_{\Phi_p}(\mathcal{S},\lambda)$ is complete linear metric space with norm defined by

$$||f||_{\Phi_p} = \inf\{c > 0 : \int_S \Phi_p(c^{-1}|f(s)|,s)\lambda(ds) \le 1\}.$$

Let $\lambda^{(1)}$ be an arbitrary probability measure on $(S,\sigma(\mathcal{S}))$ equivalent to λ. Set

$$R(r,s) := I(r\frac{d\lambda^{(1)}}{d\lambda}(s),s), \quad r > 0, \;\; s \in S,$$

where,

$$I(r,s) = \inf\{x > 0 : Q(s,[-x,x]^c) \le r\}, r > 0.$$

Define

$$\mathcal{F}_f(A) = \int_0^\infty \int_S I_{A\setminus\{0\}}(R(r,s)f(s))\lambda^{(1)}(ds)dr.$$

THEOREM 3.1. *Suppose $f \in L_{\Phi_0}$. Then $\mathcal{F}_f(A)$ is a Lévy measure.*

Proof. Since, for every $x \geq 0$ and $s \in S$,

$$Leb\{r > 0 : R(r,s) > x\} = \frac{d\lambda}{d\lambda^{(1)}}(s)Q(s,[-x,x]^c),$$

we know

$$\mathcal{F}_f(A) = \int_S \int_0^\infty I_{A\setminus\{0\}}(xf(s))Q(s,dx)\lambda(ds).$$

By [3, Proposition 2.6],

$$\exp\{iua_{f,\tau} + \int_R (e^{iux} - 1 - iuxI_{\{|x|\leq\tau\}})\mathcal{F}_f(dx)$$

is a characteristic function of $\int_S f(s)\Lambda(ds)$, where,

$$a_{f,\tau} = \int_S f(s)a(s)\lambda(ds) + \int_S \int_R (f(s)xI_{\{|f(s)x|\leq\tau\}} - f(s)xI_{\{|x|\leq\tau\}})Q(s,dx)\lambda(ds),$$

and we get

$$\begin{aligned}\int_{\{|x|\leq 1\}} x^2\mathcal{F}_f(ds) &= \int_S \int_{\{|f(s)x|\leq 1\}} |f(s)x|^2 Q(sdx)\lambda(ds) \\ &\leq \int_S |\Phi_0(|f(s)|,s)\lambda(ds) < \infty,\end{aligned}$$

$$\begin{aligned}\int_{\{|x|>1\}} \mathcal{F}_f(ds) &= \int_S \int_{\{|f(s)x|>1\}} Q(sdx)\lambda(ds) \\ &\leq \int_S |\Phi_0(|f(s)|,s)\lambda(ds) < \infty.\end{aligned}$$

Therefore, we conclude that $\mathcal{F}_f$ is a Lévy measure. □

Let $\{\xi_j\}_{j=1}^\infty$ be a sequence of i.i.d. random vectors taking values in $(S,\mathcal{S})$ with $\mathcal{L}(\xi_j) = \lambda^{(1)}$ and let Γ_j be the j-th arrival time of unit rate Poisson process. Then the following theorem holds.

THEOREM 3.2. *Suppose that $f \in L_{\Phi_p}$ for some $p \geq 1$. Then*

$$M_n(f) := \sum_{j=1}^{n} R(\Gamma_j, \xi_j) f(\xi_j) - C_f(\Gamma_n)$$

converges to $M_\infty(f)$ a.s. and in L^p as $n \to \infty$, where,

$$C_f(t) = \int_0^t \int_S R(r, s) f(s) \lambda^{(1)}(ds) dr.$$

Proof. We know that $\mathcal{F}_f$ is a Lévy measure by Theorem 3.1. Since

$$\begin{aligned} \int_{\{|x|>1\}} |x|^p \mathcal{F}_f(dx) &= \int_S \int_{\{|f(s)x|>1\}} |xf(s)|^p Q(s, dx) \lambda(ds) \\ &\leq \int_S \Phi_p(|f(s)|, s) \lambda(ds) < \infty. \end{aligned}$$

Therefore, it follows from [5, Theorem 3.1]. □

COROLLARY 3.1. *The characteristic function of $M_\infty(f)$ is*

$$\exp\{iua_f + \int_R (e^{iux} - 1 - iux) \mathcal{F}_f(dx)\},$$

where, $a_f = \int f(s) a(s) \lambda(ds)$.

4. Type G Process

We consider a large class of symmetric infinitely divisible processes, known as processes of type G, whose marginal distributions are variance mixtures of the normal distribution. The function ψ in definition 2.4 can be written as

$$\psi(y) = \int_R (1 - e^{-yt^2}) \sigma(dt), \quad y \geq 0,$$

where σ is a symmetric Lévy measure on R (i.e. satisfies $\int_R (1 \wedge t^2) \sigma(dt) < \infty$).

LEMMA 4.1. ([2]) *Define*

$$\rho(B) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma(u^{-1}B) e^{-\frac{1}{2}u^2} du, \quad B \in \mathcal{B}(R \setminus \{0\}).$$

Then ρ is a symmetric Lévy measure and

$$\psi(2^{-1}\theta^2) = \int_R (1-\cos(\theta t))\rho(dt), \quad \theta \in R. \tag{4.1}$$

Let $S = R^T$ and $\mathcal{S}$ be a sylindrical σ-field of R^T with finite measure λ. Define

$$\begin{aligned} F_A(B) &= \lambda(A)\rho(B), \\ F_f(B) &= F\{(s,x) \in R^T \times R : f(s)x \in B \setminus \{0\}\}. \end{aligned}$$

LEMMA 4.2. ([2, *Theorem* 2.1]) *Suppose that $\{X_t, t \in T\}$ is a process of type G without the Gaussian component whose f.d.d.'s are determined by* (2.2). *Then there is an independently scattered random measure Λ on $\mathcal{S}$ with characteristic function*

$$E \exp i\theta\Lambda(A) = \exp\{-\int_A \psi(2^{-1}\theta^2)\lambda(ds)\}, \quad A \in \mathcal{S}$$

such that

$$\{X_t, t \in T\} \stackrel{d}{=} \{\int_{R^T} s(t)\Lambda(ds), t \in T\},$$

in the sense of the equality of the f.d.d.'s. Moreover, for any Λ-integrable f,

$$E \exp i \int_{R^T} f d\Lambda = \exp\{-\int_{R^T} \psi(2^{-1}f^2(s))\lambda(ds)\}.$$

THEOREM 4.1. (i) *For any Λ-integrable $f : R^T \to R$,*

$$E \exp i \int_{R^T} f d\Lambda = \exp\{-\int_R (1-\cos x)\mathcal{F}_f(dx)\}.$$

(ii) *$f : R^T \to R$ is Λ-integrable if and only if $\int_R (1 \wedge x^2)\mathcal{F}_f(dx) < \infty$.*

Proof. (i) By Lemma 4.2, (4.1) and property of $\mathcal{F}_f$,

$$\begin{aligned} E \exp i \int_{R^T} f d\Lambda &= \exp\{-\int_{R^T} \psi(2^{-1}f^2(s))\lambda(ds)\} \\ &= \exp\{-\int_{R^T}\int_R (1-\cos(xf(s)))\rho(dx)\lambda(ds)\} \\ &= \exp\{-\int_R (1-\cos x)\mathcal{F}_f(dx)\}. \end{aligned}$$

(ii)

$$\begin{aligned} f\ \ is\ \ \Lambda - integrable \ \ &\Leftrightarrow \ \int_{R^T} \psi(2^{-1} f^2(s))\lambda(ds) < \infty \\ &\Leftrightarrow \ \int_{R^T}\int_R (1 \wedge x^2 f^2(s))\rho(dx)\lambda(ds) < \infty \\ &\Leftrightarrow \ \int_R (1 \wedge x^2)\mathcal{F}_f(dx) < \infty. \end{aligned}$$

□

Define

$$R(r) = \inf\{x > 0 : \rho([-x, x]^c) \le r\}.$$

Let $\lambda^{(1)}$ probability measure on R^T such that $\lambda_X \ll \lambda^{(1)}$ and let $h = \frac{d\lambda}{d\lambda^{(1)}}$.

A process $\{X_t, t \in T\}$ is of type G if and only if $\{X_t, t \in T\}$ admits the series representation

$$\{X_t, t \in T\} \stackrel{d}{=} \{\sum_{j=1}^{\infty} \zeta_j^X R_X\left(\frac{\Gamma_j^X}{h_X(V_{j,X})}\right) V_{j,X}(t),\ t \in T\} \qquad (4.2)$$

in the sense of equality of f.d.d.'s. In (4.2), the Γ_j's are the arrival times of a unit rate Poisson process, the ζ_j's are i.i.d. $N(0,1)$, the process $\{V_j(t), t \in T\}$ are i.i.d. with the common distribution related to the measure λ. Moreover, the sequences $\{\Gamma_j\}, \{\zeta_j\}, \{V_j(t), t \in T\}$ are independent.

Let $\Delta_{X_{t_{k_1}}, X_{t_{k_2}}} (t_{k_1} < t_{k_2})$ be

$$\sum_{j=1}^{\infty} R^2\left(\frac{\Gamma_j}{h_{j,X}(V_{j,X})}\right)(V_{j,X}(t_{k_1}) - V_{j,X}(t_{k_2}))^2$$

and consider $\Delta_{X_{t_1}, X_{t_2}, \cdots, X_{t_d}}$ whose (t_{k_1}, t_{t_2})-component is $\Delta_{X_{t_{k_1}}, X_{t_{k_2}}}$ and represent

$$\rho_{\Delta_{X_{t_1}, X_{t_2}, \cdots, X_{t_d}}} : (R_+^{d(d-1)/2}, \sigma(R_+^{d(d-1)/2})) \to R$$

as a Lévy measure of $\Delta_{X_{t_1}, X_{t_2}, \cdots, X_{t_d}}$.

Theorem 4.2. *Let $\{X_t, t \in T\}$ and $\{Y_t, t \in T\}$ be type G processes with Lévy measure ρ_X, ρ_Y and parameters ψ_X, ψ_Y and λ_X, λ_Y, respectively. Assume $E|X_t| < \infty$ and $E|Y_t| < \infty$. For any d, for any increasing*

Borel set A in $\sigma(R_+^{d(d-1)/2})$ and for any $t_1, t_2, \cdots, t_d \in T$,

$$\rho_{\Delta_{X_{t_1}, X_{t_2}, \cdots, X_{t_d}}}(A) \geq \rho_{\Delta_{Y_{t_1}, Y_{t_2}, \cdots, Y_{t_d}}}(A),$$

then

$$E \sup_{t \in T} X(t) \geq E \sup_{t \in T} Y(t).$$

Proof. Define

$$R_X(r) = \inf\{x > 0 : \rho_X([-x, x]^c \leq r\},$$
$$R_Y(r) = \inf\{x > 0 : \rho_Y([-x, x]^c \leq r\}.$$

Let $\lambda_X^{(1)}$ and $\lambda_Y^{(1)}$ be probability measure on R^T such that $\lambda_X \ll \lambda_X^{(1)}$ and $\lambda_Y \ll \lambda_Y^{(1)}$ and let $h_X = \dfrac{d\lambda_X}{d\lambda_X^{(1)}}$ and $h_Y = \dfrac{d\lambda_Y}{d\lambda_Y^{(1)}}$.

Then X_t and Y_t have the following series representation

$$\{X_t, t \in T\} \stackrel{d}{=} \{\sum_{j=1}^{\infty} \zeta_j^X R_X \left(\frac{\Gamma_j^X}{h_X(V_{j,X})} \right) V_{j,X}(t),\ t \in T\}, \qquad (4.3)$$

in the sense of equality of f.d.d.'s. Using a similar notation,

$$\{Y_t, t \in T\} \stackrel{d}{=} \{\sum_{j=1}^{\infty} \zeta_j^Y R_X \left(\frac{\Gamma_j^Y}{h_Y(V_{j,Y})} \right) V_{j,Y}(t),\ t \in T\}. \qquad (4.4)$$

Let $\mathcal{F}_X$ and $\mathcal{F}_Y$ be the σ-fields generated on the corresponding sample spaces by $\{\Gamma_j^X\}_{j=1}^{\infty}$ and $\{V_{j,X}(t)\}_{j=1}^{\infty}$ and by $\{\Gamma_j^Y\}_{j=1}^{\infty}$ and $\{V_{j,Y}(t)\}_{j=1}^{\infty}$, respectively. Let $\tilde{X}_t$ and $\tilde{Y}_t$ denote the right-hand sides of (4.3) and (4.4), respectively. Moreover, denoting by $E_{\mathcal{F}_X}$ ($E_{\mathcal{F}_Y}$) the conditional expectation given $\mathcal{F}_X$ ($\mathcal{F}_Y$), we obtain

$$E_{\mathcal{F}_X}(\tilde{X}(t_{k_1}) - \tilde{X}(t_{k_2}))^2 = \sum_{j=1}^{\infty} R_X^2 \left(\frac{\Gamma_j^X}{h_X(V_{j,X})} \right) (V_{j,X}(t_{k_1}) - V_{j,X}(t_{k_2}))^2,$$

$$E_{\mathcal{F}_Y}(\tilde{Y}(t_{k_1}) - \tilde{Y}(t_{k_2}))^2 = \sum_{j=1}^{\infty} R_Y^2 \left(\frac{\Gamma_j^Y}{h_Y(V_{j,Y})} \right) (V_{j,Y}(t_{k_1}) - V_{j,Y}(t_{k_2}))^2,$$

for any $t_{k_1} < t_{k_2}$. By [7, Theorem 3.1], we know that for any $A \in \sigma(R_+^{d(d-1)/2})$ and for any $t_1, t_2, \cdots, t_d \in T$,

$$\rho_{\Delta_{X_{t_1}, X_{t_2}, \cdots, X_{t_d}}}(A) \geq \rho_{\Delta_{Y_{t_1}, Y_{t_2}, \cdots, Y_{t_d}}}(A)$$

which implies $\Delta_{X_{t_1}, X_{t_2}, \cdots, X_{t_d}}$ dominates stochastically $\Delta_{Y_{t_1}, Y_{t_2}, \cdots, Y_{t_d}}$.

Thus,

$$\begin{aligned} E\sup_{t\in T} X(t) &\geq E\max_{i=1,2,\cdots,d} X(t_i) = E\max_{i=1,2,\cdots,d} \tilde{X}(t_i) \\ &\geq E\max_{i=1,2,\cdots,d} \tilde{Y}(t_i) = E\max_{i=1,2,\cdots,d} Y(t_i), \end{aligned}$$

for any $t_1, t_2, \cdots, t_d \in T$. □

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