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ANALYSIS OF THE QUADRATURE ERROR IN L_2 AND H^1 ERROR FOR THE h VERSION OF THE FEM

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Abstract. We describe some results for the h version which pertain to the questions on numerical quadrature. We also present an example that illustrates the rate of convergence predicted for linear elements under certain quadrature schemes in [2], [4], [5].

We briefly describe some results for the h version which pertain to the questions on numerical quadrature. We also present an example that illustrates the rate of convergence predicted for linear elements under certain quadrature schemes in [2], [4], [5].

First of all, the basic rule to ensure optimal convergence in the H^1 norm is to ensure that for piecewise polynomials of degree p (either on a one-dimensional grid or on a quasiuniform family of triangular meshes), the quadrature scheme chosen is exact for all polynomials of degree $\leq 2p-2$. This implies, for instance, that for $p = 1$ (linear elements), the left end-point rule is sufficient in the one-dimensional case and that the mid-point rule (for triangles) is sufficient for triangular meshes. For rectangular meshes, using $\mathcal{Q}_p$ elements, the rule must be exact for the space $\mathcal{Q}_{2p-1}(\hat{K})$ *and* the union $\cup_{l=1}^{L_p}\{\hat{b}_l^p\}$ must contain a $\mathcal{Q}_p(\hat{K}) \cap \mathcal{P}_{2p-1}(\hat{K})$–unisolvent subset. This means , for instance, that in two and higher dimensions, $p+1$ G-L points would have to be used in each direction (the same minimum as for the p version), to ensure the unisolvency.

Suppose now that instead of the H^1 norm, a lower order norm is of interest. Then, in [2], it has been shown that if piecewise polynomials

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of degree $\leq p$ are used (on a one-dimensional mesh), then

$$\|u - \tilde{u}_h\|_{l,\Omega} = O(h^{p+1-l}), \tag{1}$$

for $2 - p \leq l \leq 1$, when the quadrature rule has precision $2p - 2$. Moreover, (1) will also hold for $l = 1 - p$, provided the rule has an *extra* order of precision, ie, it is exact for polynomials of degree $\leq 2p - 1$ (see also [4] and [5]). This latter condition is only stated to be *sufficient* in the above references. In [1], the necessity of such a condition has been shown in the context of guaranteeing convergence of eigenvalues for the eigenvalue problem.

In this section, we will show the necessity of requiring this extra degree of precision for the case that we have *mapped* linear elements. The results in [2], [4], [5] pertain to the case of *non-constant coefficients*. In our example, however, we choose the coefficient to be a constant but map the elements instead.

We consider the boundary value problem $-u'' = f$ on $I = [0, 1]$, with the boundary conditions $u(0) = 0, u'(1) = 1$. Suppose $f = 0$, so that the exact solution is $u(x) = x$. We consider the h version on I with a uniform mesh given by $x_i = ih, \quad i = 0, 1, \ldots, n, \quad$ where $h = 1/n$. Assume that the i^{th} subinterval $I_i = [(i-1)h, ih]$ is the image of the reference interval $\hat{I} = [0, 1]$ under the mapping

$$x = (i-1)h + hg(\xi), \tag{2}$$

where

$$g(\xi) = \xi + \frac{h}{2}(\xi^2 - \xi). \tag{3}$$

Suppose we use linear elements on $\hat{I}$, so that the basis functions on the reference element are $\hat{\psi}_A(\xi) = \xi$ and $\hat{\psi}_B(\xi) = 1 - \xi$. On I_i, the corresponding basis functions will be

$$\psi_i(x(\xi)) = \hat{\psi}_A(\xi) = \xi, \quad \psi_{i-1}(x(\xi)) = \hat{\psi}_B(\xi) = 1 - \xi.$$

Let us compute the entries of the stiffness matrix when exact integration is used. We have

$$\begin{aligned} b &= \int_{(i-1)h}^{ih} \psi_i' \psi_i' \, dx = \int_0^1 \frac{\hat{\psi}_A'(\xi)\hat{\psi}_A'(\xi)}{hg'(\xi)} \, d\xi \\ &= \frac{1}{h} \int_0^1 (1 - \frac{h}{2}(2\xi - 1) + O(h^2)) \, d\xi = \frac{1}{h}(1 + \beta h^2), \end{aligned} \tag{4}$$

since $\int_0^1 (2\xi - 1) = 0$ (this is why we selected the special form in (3)). The other terms may be similarly calculated.

Suppose K_1 is the stiffness matrix when $g(\xi) = \xi$ in (2). Then for $g(\xi)$ as in (3), the above calculation shows that we get the stiffness matrix

$$K = K_1(1 + \beta h^2). \tag{5}$$

The solution that we get is

$$u^h(x) = \sum_{i=1}^{n} c_i \, \psi_i(x), \tag{6}$$

where $\vec{c} = (c_1, c_2, \ldots, c_n)^T$ satisfies

$$K\vec{c} = (0, 0, \ldots, 0, 1)^T. \tag{7}$$

Since the solution of (7) when $K = K_1$ is simply $c_i = ih$, it follows that the solution when K is given by (5) is

$$c_i = ih(1 + \beta \, h^2)^{-1}. \tag{8}$$

Suppose now that we use numerical integration with a rule that has $O(h^\alpha)$ accuracy, so that instead of (4), we obtain

$$h(\tilde{b} - b) = O(h^\alpha)$$

i.e.,

$$b = \frac{1}{h}(1 + \gamma \, h^\alpha + \beta \, h^2).$$

Then for $\alpha \leq 2$, the solution corresponding to (6) and (8) will be

$$\tilde{u}^h(x) = \sum_{i=1}^{n} \tilde{c}_i \, \psi_i(x), \qquad \tilde{c}_i = ih(1 + \tilde{\beta} \, h^\alpha)^{-1}. \tag{9}$$

We now calculate the H^1 error with $\tilde{u}^h$, u^h. Consider the i^{th} subinterval I_i. Let us define U_i, $\tilde{U}_i^h$ on $\hat{I}$ such that for $x \in I_i$,

$$u(x) = U_i(\xi), \qquad \tilde{u}^h(x) = \tilde{U}_i^h(\xi).$$

Then

$$U_i(\xi) = (i-1)h + hg(\xi),$$

$$\tilde{U}_i^h(\xi) = ((i-1)h + h\xi)(1 + \tilde{\beta} h^\alpha)^{-1},$$

and we get with $\tilde{e}^h = u - \tilde{u}^h$,

$$|\tilde{e}^h|^2_{1,I_i} = \int_0^1 \frac{(U_i'(\xi) - \tilde{U}_i^{h\prime}(\xi))^2}{hg'(\xi)}\, d\xi$$
$$= h\int_0^1 \frac{(g'(\xi) - (1+\tilde{\beta} h^\alpha)^{-1})^2}{g'(\xi)}\, d\xi$$
$$= h(A\, h^{2\alpha} + B\, h^2 + C\, h^{\alpha+1}),$$

using (3). Summing from $i = 1$ to n gives, for $\alpha \geq 1$,

$$|\tilde{e}^h|^2_{1,I} \;\approx\; Ch^2,$$

so that the H^1 error is $O(h)$ if the integration rule gives $O(h)$ accuracy. Taking $\tilde{\beta} = 0$ in the above shows that using exact integration, the H^1 error is also $O(h)$, so that $O(h)$ accuracy is sufficient for an optimal H^1 error.

Next, we calculate the L_2 error. Setting

$$(1 + \tilde{\beta}\, h^\alpha)^{-1} \;\approx\; 1 - \tilde{\beta}\, h^\alpha,$$

we have

$$|\tilde{e}^h|^2_{0,I_i} = h\int_0^1 (U_i(\xi) - \tilde{U}_i^h(\xi))^2 g'(\xi)\, d\xi$$
$$= h^3 \int_0^1 \{\tilde{\beta}(i-1)h^\alpha$$
$$+ (g(\xi) - \xi(1-\tilde{\beta}h^\alpha))\}^2 g'(\xi)\, d\xi$$
$$+ \text{ lower order terms.}$$

Using the fact that

$$g(\xi) - \xi = \frac{h}{2}(\xi^2 - \xi),$$

we have

$$|\tilde{e}^h|^2_{0,I_i} \;\approx\; h^3(i^2 D h^{2\alpha} + E\, h^2).$$

Summing from $i = 1$ to n, we have for $1 \leq \alpha \leq 2$,

$$|\tilde{e}^h|^2_{0,I} \;\approx\; Ch^{2\alpha},$$

so that the L_2 error behaves like Ch^α. Using the same argument above with $\tilde{\beta} = 0$ gives $|e^h|_{0,I} = Ch^2$ for $e^h = u - u^h$. Hence, to preserve this (optimal) rate of convergence, we must have $\alpha = 2$ rather than 1 in the quadrature estimate.

FIGURE 1. H^1 and L^2 errors with the h version.

In terms of degree of precision of the rule used, this shows that we need (for the one-dimensional case), a rule that is exact for polynomials of degree ≤ 1 $(= 2p-1)$ rather than ≤ 0 $(= 2p-2)$. For example, the left (or right) end-point rule is not sufficient, we would need either the mid-point or trapezoidal rule.

In Figure 1, we present the results of numerical computation using the above quadrature rules. If we use middle point rule (ie, $\int h(x)dx = h(\frac{1}{2})$) to calculate stiffness matrix, then the order is $0(h^2)$ and if we use left point rule (ie, $\int h(x)dx = h(0)$) the order is $0(h)$. The condition $\int_0^1 \phi(\xi)d\xi = 0$, we choose $\phi(\xi) = \xi - \frac{1}{2}$. First, the curves (a) and (b) respectively represent the L_2 and H^1 errors on a log-log scale, which are obtained by using the left end-point rule. As expected, the order in both cases is $O(h)$, showing that the H^1 error is optimal but the L_2 error is not. On the other hand, using the mid-point rule (dotted lines (c) and (d)) shows that the H^1 error is $O(h)$ (in (c)) and the L_2 error is $O(h^2)$ (in (d)), both of which are optimal.

References

[1] U. Banerjee. *A note on the effect of numerical quadrature in finite element eigenvalue approximation.* Technical report, Department of Mathematics, Syracuse University, 1991.

[2] U. Banerjee and J. E. Osborn. *Estimation of the effect of numerical integration in finite element eigenvalue approximation.* Numer. Math. **56** (1990), 735–762.

[3] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems.*, North-Holland, Amsterdam, 1978

[4] P. G. Ciarlet and P. A. Raviart. *The combined effect of curved boundaries and numerical integration in isoparametric finite element methods. The Mathematical Foundations of the Finite Element Method*, A. K. Aziz, Ed., (1972), 409–474.

[5] Jr. J. Douglas and T. Dupont. *Galerkin approximation for the two point boundary value problem using continuous, piecewise polynomial spaces.* Numer. Math. **22** (1974), 99–109.

[6] Claes Johnson. *Numerical Solution of Partial Differential Equations by the Finite Element Method.*, Cambridge University Press, New-York, 1987

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