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FUZZY R-CLUSTER AND FUZZY R-LIMIT POINTS

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Abstract. In this paper, we introduce the notions of fuzzy r-cluster and fuzzy r-limit points in smooth fuzzy topological spaces and investigate some of their properties.

1. Introduction and preliminaries

A.P. Sostak [11] introduced the smooth fuzzy topology as an extension of Chang's fuzzy topology [1]. It has been developed in many directions [2,5,6]. Pu and Liu [10] introduced the notions of fuzzy nets and Q-neighborhoods and established the convergence theory in fuzzy topological spaces. In 1994, Chen and Cheng [3] introduced the concepts of fuzzy cluster and fuzzy limit points in fuzzy topological spaces with respect to R-neighborhood instead of Q-neighborhood.

In this paper, we introduce the concept of fuzzy r-cluster and fuzzy r-limit points in a smooth fuzzy topological space as an extension of [10] and investigate some of their properties and give an example of those.

Throughout this paper, let X be a nonempty set, $I = [0,1]$ and $I_0 = (0,1]$. A *fuzzy point* x_t for $t \in I_0$ is an element of I^X such that, for $y \in X$,

$$x_t(y) = \begin{cases} t \text{ if } y = x, \\ 0 \text{ if } y \neq x. \end{cases}$$

The set of all fuzzy points in X is denoted by $Pt(X)$. Let $x_t \in Pt(X)$ and $\lambda, \mu \in I^X$. $x_t \in \lambda$ iff $t \leq \lambda(x)$ for $x \in X$. λ is *quasi-coincident* with μ, denoted by $\lambda\ q\ \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If λ is not quasi-coincident with μ, we denote $\lambda\ \overline{q}\ \mu$. All the other notations and the other definitions are standard in fuzzy set theory.

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Definition 1.1 ([11]). A function $\mathcal{T} : I^X \to I$ is called a *smooth fuzzy topology* on X if it satisfies the following conditions:

(O1) $\mathcal{T}(\tilde{0}) = \mathcal{T}(\tilde{1}) = 1$, where $\tilde{0}(x) = 0$ and $\tilde{1}(x) = 1$ for all $x \in X$.

(O2) $\mathcal{T}(\mu_1 \wedge \mu_2) \geq \mathcal{T}(\mu_1) \wedge \mathcal{T}(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$.

(O3) $\mathcal{T}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(\mu_i)$, for any $\{\mu_i\}_{i\in\Gamma} \subset I^X$.

The pair $(X, \mathcal{T})$ is called a *smooth fuzzy topological space.*

Theorem 1.2 ([2]). *Let $(X, \mathcal{T})$ be a smooth fuzzy topological space. For each $r \in I_0$ and $\lambda \in I^X$, we define a fuzzy closure operator $C_{\mathcal{T}} : I^X \times I_0 \to I^X$ as follows:*

$$C_{\mathcal{T}}(\lambda, r) = \bigwedge\{\rho \in I^X \mid \lambda \leq \rho,\ \mathcal{T}(\tilde{1} - \rho) \geq r\}.$$

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following properties:

(1) $C_{\mathcal{T}}(\tilde{0}, r) = \tilde{0}$.

(2) $\lambda \leq C_{\mathcal{T}}(\lambda, r)$.

(3) $C_{\mathcal{T}}(\lambda, r) \vee C_{\mathcal{T}}(\mu, r) = C_{\mathcal{T}}(\lambda \vee \mu, r)$.

(4) $C_{\mathcal{T}}(\lambda, r) \leq C_{\mathcal{T}}(\lambda, s)$, *if* $r \leq s$.

(5) $C_{\mathcal{T}}(C_{\mathcal{T}}(\lambda, r), r) = C_{\mathcal{T}}(\lambda, r)$.

Definition 1.3 ([7]). Let $\lambda, \mu \in I^X$. Define the *fuzzy quasi-difference* of λ and μ, denoted by $\lambda \setminus \mu$, as

$$(\lambda \setminus \mu)(x) = \begin{cases} \lambda(x), & \text{if } \mu(x) = 0, \\ 0, & \text{if } \lambda(x) \geq \mu(x) > 0, \\ \lambda(x), & \text{if } \lambda(x) < \mu(x). \end{cases}$$

Notation 1.4. Let $(X, \mathcal{T})$ be a smooth fuzzy topological space, $\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. We denote

$$\mathcal{N}(x_t, r) = \{\mu \in I^X \mid x_t\ q\ \mu,\ \mathcal{T}(\mu) \geq r\}.$$

Definition 1.5 ([7]). Let $(X, \mathcal{T})$ be a smooth fuzzy topological space, $\lambda \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$.

(1) x_t is called an *fuzzy r-adherent point* of λ if for every $\mu \in \mathcal{N}(x_t, r)$, we have $\mu\ q\ \lambda$.

(2) x_t is called a *fuzzy r-accumulation point* of λ if for every $\mu \in \mathcal{N}(x_t, r)$, we have $\mu \, q \, (\lambda \setminus x_t)$.

(3) Define the *fuzzy r-derived set* of λ, denote by $D_{\mathcal{T}}(\lambda, r)$, as

$$D_{\mathcal{T}}(\lambda, r) = \bigvee \{x_t \in Pt(X) \mid x_t \text{ is a fuzzy r-accumulation point of } \lambda\}.$$

Theorem 1.6 ([7]). *Let $(X, \mathcal{T})$ be a smooth fuzzy topological space. For each $\lambda \in I^X$ and $r \in I_0$, we have*

$$C_{\mathcal{T}}(\lambda, r) = \bigvee \{x_t \in Pt(X) \mid x_t \textit{ is a fuzzy r-adherent point of } \lambda\}.$$

Theorem 1.7 ([7]). *Let $(X, \mathcal{T})$ be a smooth fuzzy topological space. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, the following properties hold:*

(1) $D_{\mathcal{T}}(\lambda, r) \leq C_{\mathcal{T}}(\lambda, r)$.

(2) $C_{\mathcal{T}}(\lambda, r) = \lambda \vee D_{\mathcal{T}}(\lambda, r)$.

(3) $C_{\mathcal{T}}(\lambda, r) = \lambda$ *iff* $D_{\mathcal{T}}(\lambda, r) \leq \lambda$.

(4) *If* $r \leq s$, *then* $D_{\mathcal{T}}(\lambda, r) \leq D_{\mathcal{T}}(\lambda, s)$.

(5) $D_{\mathcal{T}}(\lambda \vee \mu, r) \leq D_{\mathcal{T}}(\lambda, r) \vee D_{\mathcal{T}}(\mu, r)$.

2. Fuzzy r-cluster points and r-limit points

Definition 2.1. Let D be a directed set. A function $S : D \to Pt(X)$ is called a *fuzzy net*. Let $\lambda \in I^X$. We say S is a *fuzzy net in* λ if $S(n) \in \lambda$ for every $n \in D$.

Using Notation 1.4 , we can define the followings:

Definition 2.2. Let $(X, \mathcal{T})$ be a smooth fuzzy topological space, $\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$.

(1) x_t is called a *fuzzy r-cluster point* of S, denoted by $S \overset{r}{\infty} x_t$, if for every $\mu \in \mathcal{N}(x_t, r)$, S is frequently quasi-coincident with μ, i.e, for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0) \, q \, \mu$.

(2) x_t is called a *fuzzy r-limit point* of S, denoted by $S \overset{r}{\to} x_t$, if for every $\mu \in \mathcal{N}(x_t, r)$, S is eventually quasi-coincident with μ, i.e,

there exists $n_0 \in D$ such that for each $n \in D$ with $n \geq n_0$, we have $S(n)\ q\ \mu$. We denote

$$clu_{\mathcal{T}}(S,r) = \bigvee\{x_t \in Pt(X) \mid x_t \text{ is a fuzzy r-cluster point of } S\},$$

$$lim_{\mathcal{T}}(S,r) = \bigvee\{x_t \in Pt(X) \mid x_t \text{ is a fuzzy r-limit point of } S\}.$$

DEFINITION 2.3. Let $(X,\mathcal{T})$ be a smooth fuzzy topological space. Let $S : D \to Pt(X)$ and $W : E \to Pt(X)$ be two fuzzy nets. W is called a *subnet* of S if there exists a function $N : E \to D$, called by a *cofinal selection* on S, such that

(1) $W = S \circ N$;
(2) For every $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$ for $m \geq m_0$.

THEOREM 2.4. *Let $(X,\mathcal{T})$ be a smooth fuzzy topological space. Let $S : D \to Pt(X)$ fuzzy net and $W : E \to Pt(X)$ a subnet of S. For $r,s \in I_0$, the following properties hold:*

(1) *If $S \overset{r}{\to} x_t$, then $S \overset{r}{\infty}\ x_t$.*
(2) *$lim_{\mathcal{T}}(S,r) \leq clu_{\mathcal{T}}(S,r)$.*
(3) *If $S \overset{r}{\infty}\ x_t$ and $x_t \geq x_s$, then $S \overset{r}{\infty}\ x_s$.*
(4) *If $S \overset{r}{\to} x_t$ and $x_t \geq x_s$, then $S \overset{r}{\to} x_s$.*
(5) *$S \overset{r}{\infty}\ x_t$ iff $x_t \in clu_{\mathcal{T}}(S,r)$.*
(6) *$S \overset{r}{\to} x_t$ iff $x_t \in lim_{\mathcal{T}}(S,r)$.*
(7) *If $S \overset{r}{\to} x_t$, then $W \overset{r}{\to} x_t$.*
(8) *$lim_{\mathcal{T}}(S,r) \leq lim_{\mathcal{T}}(W,r)$.*
(9) *If $W \overset{r}{\infty}\ x_t$, then $S \overset{r}{\infty}\ x_t$.*
(10) *$clu_{\mathcal{T}}(W,r) \leq clu_{\mathcal{T}}(S,r)$.*

Proof. (1)and (2) are clear.

(3) For every $\mu \in \mathcal{N}(x_s,r)$, since $x_s \leq x_t$, then $\mu \in \mathcal{N}(x_t,r)$. Since $S \overset{r}{\infty}\ x_t$, for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0)\ q\ \mu$. Hence $S \overset{r}{\infty}\ x_s$.

(4) It is similar to (3).

(5) $(\Rightarrow)$ It is clear.

($\Leftarrow$) Let $x_t \in clu_{\mathcal{T}}(S,r)$ and $\mu \in \mathcal{N}(x_t,r)$. Since $x_t\ q\ \mu$ and $clu_{\mathcal{T}}(S,r)(x) \geq t$, we have

$$\mu(x) + clu_{\mathcal{T}}(S,r)(x) \geq \mu(x) + t > 1.$$

From the definition of $clu_{\mathcal{T}}(S,r)$, there exists a fuzzy r-cluster point $x_s \in Pt(X)$ of S such that

$$\mu(x) + clu_{\mathcal{T}}(S,r)(x) \geq \mu(x) + s > 1.$$

Thus $\mu \in \mathcal{N}(x_s,r)$. Since x_s is a fuzzy r-cluster point of S , for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0)\ q\ \mu$. Hence $S \overset{r}{\infty}\ x_t$.

(6) It is similar to (5).

(7) For every $\mu \in \mathcal{N}(x_t,r)$, since $S \overset{r}{\rightarrow} x_t$, there exists $n_0 \in E$ such that for all $n \geq n_0$, $S(n)\ q\ \mu$. Let $N : E \rightarrow D$ be a cofinal selection on S. Then for $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$ for all $m \geq m_0$. Thus $W(m) = S(N(m))\ q\ \mu$ for $m \geq m_0$. Therefore, $W \overset{r}{\rightarrow} x_t$.

(8) From (7), it is clear.

(9) Suppose that $W \overset{r}{\infty}\ x_t$ and $n \in D$. If $N : E \rightarrow D$ is a cofinal selection on S, then there exists $m \in E$ such that $N(k) \geq n$ for $k \geq m$. Since $W \overset{r}{\infty}\ x_t$, for every $\mu \in \mathcal{N}(x_t,r)$, there exists $m_0 \in E$ such that $m_0 \geq m$ and $W(m_0)\ q\ \mu$. We let $n_0 = N(m_0)$. Then $n_0 \geq n$ and since $S(n_0) = W(m_0)$, we have $S(n_0)\ q\ \mu$.

(10) From (9), it is clear. □

Theorem 2.5. *Let $(X,\mathcal{T})$ be a smooth fuzzy topological space and $x_t \in Pt(X)$ and $r \in I_0$. For every fuzzy net S , $S \overset{r}{\rightarrow} x_t$ iff $W \overset{r}{\infty}\ x_t$, for every fuzzy subnet W of S.*

Proof. ($\Rightarrow$) From Theorem 2.4(7), $S \overset{r}{\rightarrow} x_t$ implies $W \overset{r}{\rightarrow} x_t$. From Theorem 2.4 (9), $W \overset{r}{\rightarrow} x_t$ implies $W \overset{r}{\infty}\ x_t$.

($\Leftarrow$) Suppose x_t is not a fuzzy r-limit point x_t of S. Then there exists $\mu \in \mathcal{N}(x_t,r)$ satisfying the followings: for each $n \in D$, there exists $N(n) \in D$ such that $N(n) \geq n$ and $S(N(n))\ \overline{q}\ \mu$. We can define $N : D \rightarrow D$. For each $m \geq n$, we have $N(m) \geq m \geq n$. Hence N is a cofinal selection on S. So, $W = S \circ N$ is a fuzzy subnet of S. Since for $\mu \in \mathcal{N}(x_t, r)$ and for each $n \in D$, $W(n) = S(N(n))\overline{q}\ \mu$, x_t is not fuzzy r-cluster point of W. □

THEOREM 2.6. *Let $(X, \mathcal{T})$ be a smooth fuzzy topological space and $x_t \in Pt(X)$ and $r \in I_0$. For every fuzzy net $S : D \to Pt(X)$, we have $S \overset{r}{\infty} x_t$ iff S has a fuzzy subnet W such that $W \overset{r}{\to} x_t$.*

Proof. ($\Rightarrow$) Let $E = D \times \mathcal{N}(x_t, r) = \{(m, \lambda) \mid m \in D, \lambda \in \mathcal{N}(x_t, r)\}$. Define a relation on E by

$$\forall (m, \lambda), (n, \mu) \in E, \ \ (m, \lambda) \leq (n, \mu) \Leftrightarrow \ \ m \leq n, \lambda \geq \mu.$$

For each $(m, \lambda), (n, \mu) \in E$, we have $\lambda, \mu \in \mathcal{N}(x_t, r) \ \Rightarrow \ \lambda \wedge \mu \in \mathcal{N}(x_t, r)$ and there exists $k \in D$ such that $m \leq k$ and $n \leq k$. Hence there exists $(k, \lambda \wedge \mu) \in E$ such that $(m, \lambda) \leq (k, \lambda \wedge \mu)$ and $(n, \mu) \leq (k, \lambda \wedge \mu)$. So, E is a directed set. For each $(n, \mu) \in E$, since $S \overset{r}{\infty} x_t$, there exists $N(n, \mu) \in D$ such that $N(n, \mu) \geq n$ and $S(N(n, \mu))\ q\ \mu$. So, we can define $N : E \to D$. For each $n_0 \in D$, since $S \overset{r}{\infty} x_t$, for $\mu_0 \in \mathcal{N}(x_t, r)$, there exists $(n_0, \mu_0) \in E$ such that $N(n_0, \mu_0) \geq n_0$. Hence for every $(n, \mu) \geq (n_0, \mu_0)$, since $n \geq n_0$, we have $N(n, \mu) \geq n \geq n_0$. Therefore N is a cofinal selection on S. So, $W = S \circ N$ is a fuzzy subnet of S. Now we show that $W \overset{r}{\to} x_t$. For each $\mu_0 \in \mathcal{N}(x_t, r)$, since $S \overset{r}{\infty} x_t$, for $n_0 \in D$, there exists $N(n_0, \mu_0) \in D$ such that $S(N(n_0, \mu_0))\ q\ \mu_0$. Hence for every $(n, \mu) \geq (n_0, \mu_0)$, $S(N(n, \mu))\ q\ \mu$ implies $S(N(n, \mu))\ q\ \mu_0$ because $\mu \leq \mu_0$. So, $W \overset{r}{\to} x_t$.

($\Leftarrow$) From Theorem 2.4(1), $W \overset{r}{\to} x_t$ implies $W \overset{r}{\infty} x_t$. From Theorem 2.4(9), $W \overset{r}{\infty} x_t$ implies $S \overset{r}{\infty} x_t$. □

THEOREM 2.7. *Let $(X, \mathcal{T})$ be a smooth fuzzy topological space and $x_t \in Pt(X)$ and $r \in I_0$. Then the following statements are equivalent.*

(1) $x_t \in C_{\mathcal{T}}(\lambda, r)$

(2) There exists a fuzzy net S in λ such that $S \overset{r}{\infty} x_t$.

(3) There exists a fuzzy net S in λ such that $S \overset{r}{\to} x_t$.

Proof. (1)$\Rightarrow$(2) Define a relation on $\mathcal{N}(x_t, r)$ by,

$$\nu \preceq \omega \ \text{ iff } \ \omega \leq \nu, \ \ \forall \nu, \omega \in \mathcal{N}(x_t, r).$$

Then $(\mathcal{N}(x_t, r), \preceq)$ is a directed set.

For each $\mu \in \mathcal{N}(x_t, r)$, since $x_t \in C_{\mathcal{T}}(\lambda, r)$, we have

$$C_{\mathcal{T}}(\lambda, r)(x) + \mu(x) \geq t + \mu(x) > 1.$$

From Theorem 1.6, there exists fuzzy r-adherent point x_s of λ such that

$$C_{\mathcal{T}}(\lambda, r)(x) + \mu(x) \geq s + \mu(x) > 1.$$

Since x_s is a fuzzy r-adherent point of λ and $\mu \in \mathcal{N}(x_s, r)$, we have $\lambda \; q \; \mu$. Then there exist $y \in X$ and $m \in I_0$ such that

$$\lambda(y) + \mu(y) \geq m + \mu(y) > 1.$$

Hence $y_m \in \lambda$ and $\mu \in \mathcal{N}(y_m, r)$. Define a directed set $(\mathcal{N}(x_t, r), \preceq)$ by

$$\nu \preceq \mu \ \text{ iff } \ \mu \leq \nu.$$

For each $\mu \in \mathcal{N}(x_t, r)$, we can define a fuzzy net $S : \mathcal{N}(x_t, r) \to Pt(X)$ by $S(\mu) = y_m$. Then $S(\mu) \; q \; \mu$ and $S(\mu) \in \lambda$.

Now we will show that $S \overset{r}{\infty} x_t$. Let $\mu \in \mathcal{N}(x_t, r)$. Then for every $\nu \in \mathcal{N}(x_t, r)$, we have $\mu \wedge \nu \in \mathcal{N}(x_t, r)$ and $S(\mu \wedge \nu) \; q \; (\mu \wedge \nu)$. Thus $\nu \preceq \mu \wedge \nu$ and $S(\mu \wedge \nu) \; q \; \mu$.

(2) $\Rightarrow$ (1) If there exists a fuzzy net S in λ such that $S \overset{r}{\infty} x_t$, for each $\mu \in \mathcal{N}(x_t, r)$ and for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $S(n_0) \; q \; \mu$. Since $S(n_0) \in \lambda$, $S(n_0) \; q \; \mu$ implies $\lambda \; q \; \mu$. Hence x_t is fuzzy r-adherent point of λ, that is, $x_t \in C_{\mathcal{T}}(\lambda, r)$.

(2)$\Rightarrow$(3) It is easily proved from Theorem 2.6.

(3)$\Rightarrow$(2) It is easily proved from Theorem 2.4(1). □

Theorem 2.8. *Let $(X, \mathcal{T})$ be a smooth fuzzy topological space and $x_t \in Pt(X)$ and $r \in I_0$. Then the following statements are equivalent.*

(1) $C_{\mathcal{T}}(\lambda, r) = \lambda$.

(2) For every fuzzy net S in λ and $x_t \in Pt(x)$, if $S \overset{r}{\infty} x_t$, then $x_t \in \lambda$.

(3) For every fuzzy net S in λ and $x_t \in Pt(x)$, if $S \overset{r}{\to} x_t$, then $x_t \in \lambda$.

Proof. (1)⇒(2) Suppose that there exists a fuzzy net S in λ such that $S \overset{r}{\infty} x_t$ but $x_t \notin \lambda$. From Theorem 2.7, $x_t \in C_{\mathcal{T}}(\lambda, r)$. Hence $C_{\mathcal{T}}(\lambda, r)(x) \geq t > \lambda(x)$. Thus $C_{\mathcal{T}}(\lambda, r) \neq \lambda$.

(2) ⇒ (1) If $x_t \in C_{\mathcal{T}}(\lambda, r)$, by Theorem 2.7, there exists a fuzzy net S in λ such that $S \overset{r}{\infty} x_t$. Hence $x_t \in \lambda$ from (2). Thus $C_{\mathcal{T}}(\lambda, r) \leq \lambda$. From Theorem 1.2(2), we have $C_{\mathcal{T}}(\lambda, r) = \lambda$.

(1)⇒(3) and (3)⇒(1) are similarly proved. □

Using Definition 1.5 and Theorem 2.7, we can easily prove the following corollary.

Corollary 2.9. *Let $(X, \mathcal{T})$ be a smooth fuzzy topological space and $x_t \in Pt(X)$ and $r \in I_0$. Then the following statements are equivalent.*

(1) $x_t \in D_{\mathcal{T}}(\lambda, r)$

(2) There exists a fuzzy net S in $\lambda \setminus x_t$ such that $S \overset{r}{\infty} x_t$.

(3) There exists a fuzzy net S in $\lambda \setminus x_t$ such that $S \overset{r}{\to} x_t$.

Example 2.10. Let $X = \{x, y\}$ be set. Define $\mu \in I^X$ as follows:

$$\mu(x) = 0.3,\ \mu(y) = 0.4.$$

We define a smooth fuzzy topology $\mathcal{T} : I^X \to I$ as follows:

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Let N be a natural number set. Define a fuzzy net $S : N \to Pt(X)$ by

$$S(n) = x_{a_n},\ a_n = 0.6 + (-1)^n 0.2 + \frac{0.2}{n}.$$

We can show $clu_{\mathcal{T}}(S, \frac{1}{2}) = \tilde{1}$ from (1) to (4)

(1) x_t for $t \leq 0.7$ is a fuzzy $\frac{1}{2}$-cluster point of S, for $\tilde{1} \in \mathcal{N}(x_t, \frac{1}{2})$ and for all $n \in N$,we have $S(n)\ q\ \tilde{1}$.

(2) x_t for $0.7 < t$ is a fuzzy $\frac{1}{2}$-cluster point of S, for $\tilde{1}, \mu \in \mathcal{N}(x_t, \frac{1}{2})$ and for all $n \in N$, there exists $2n \in N$ such that $2n \geq n$, $S(2n) = x_{0.8+\frac{0.2}{n}}\ q\ \mu$ and $S(2n) = x_{0.8+\frac{0.2}{n}}\ q\ \tilde{1}$.

(3) y_s for $s \leq 0.6$ is a fuzzy $\frac{1}{2}$-cluster point of S, for $\tilde{1} \in \mathcal{N}(y_s, \frac{1}{2})$ and for all $n \in N$,we have $S(n)\ q\ \tilde{1}$.

(4) y_s for $0.6 < s$ is a fuzzy $\frac{1}{2}$-cluster point of S, for $\tilde{1}, \mu \in \mathcal{N}(y_s, \frac{1}{2})$ and for all $n \in N$, there exists $2n \in N$ such that $2n \geq n$, $S(2n) = x_{0.8+\frac{0.2}{n}}\ q\ \tilde{1}$ and $S(2n) = x_{0.8+\frac{0.2}{n}}\ q\ \mu$.

We can show $lim_{\mathcal{T}}(S, \frac{1}{2}) = \tilde{1} - \mu$ from (5) to (8).

(5) x_t for $t \leq 0.7$ is a fuzzy $\frac{1}{2}$-limit point of S, for $\tilde{1} \in \mathcal{N}(x_t, \frac{1}{2})$ and for all $n \in N$,we have $S(n)\ q\ \tilde{1}$.

(6) x_t for $0.7 < t$ is not a fuzzy $\frac{1}{2}$-limit point of S, there exists $\mu \in \mathcal{N}(x_t, \frac{1}{2})$ such that for all $n \in N$, there exists $2n+1 \in N$ such that $2n+1 \geq n$ and $S(2n+1) = x_{0.4+\frac{0.2}{2n+1}}\ \overline{q}\ \mu$.

(7) y_s for $s \leq 0.6$ is a fuzzy $\frac{1}{2}$-limit point of S, for $\tilde{1} \in \mathcal{N}(y_s, \frac{1}{2})$ and for all $n \in N$,we have $S(n)\ q\ \tilde{1}$.

(8) y_s for $0.6 < s$ is not a fuzzy $\frac{1}{2}$-limit point of S, there exists $\mu \in \mathcal{N}(x_t, \frac{1}{2})$ such that for all $n \in N$, there exists $2n+1 \in N$ such that $2n+1 \geq n$ and $S(2n+1) = x_{0.4+\frac{0.2}{2n+1}}\ \overline{q}\ \mu$. Hence $lim_{\mathcal{T}}(S, \frac{1}{2}) = \tilde{1} - \mu$.

Define $\Psi : N \to N$ by $\Psi(n) = 2n+1$. Then Ψ is a cofinal selection on S. W is a subnet of S. Since $W(n) = S \circ \Psi(n) = S(2n+1) = x_{0.4+\frac{0.2}{2n+1}}$, as the above methods, we can obtain $clu_{\mathcal{T}}(W, \frac{1}{2}) = \tilde{1} - \mu$. □

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