Kangweon-Kyungki Math. Jour. **8** (2000), No. 1, pp. 73–86

GENERALIZED SOBOLEV SPACES OF EXPONENTIAL TYPE

Sungjin Lee

Abstract. We study the Sobolev spaces to the generalized Sobolev spaces $H^s_{\mathcal{G}}$ of exponential type based on the Silva space $\mathcal{G}$ and investigate its properties such as imbedding theorem and structure theorem. In fact, the imbedding theorem says that for $s > 0$ $u \in H^s_{\mathcal{G}}$ can be analytically continued to the set $\{z \in \mathbb{C}^n \,|\, |\mathrm{Im}\, z| < s\}$. Also, the structure theorem means that for $s > 0$ $u \in H^{-s}_{\mathcal{G}}$ is of the form

$$u = \sum_{\alpha} \frac{s^{|\alpha|}}{\alpha!} D^\alpha g_\alpha$$

where g_α's are square integrable functions for $\alpha \in \mathbb{N}_0^n$.

Moreover, we introduce a classes of symbols of exponential type and its associated pseudo-differential operators of exponential type, which naturally act on the generalized Sobolev spaces of exponential type.

Finally, a generalized Bessel potential is defined and its properties are investigated.

1. Introduction

The Sobolev space H^s serves as a very useful tool in the theory of partial differential equations, which is defined as follows.

Definition 1.1. A tempered distribution $u \in \mathcal{S}'$ belongs to the Sobolev space $H^s = H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ if $\hat{u}(\xi)$ is a function and

$$\int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty. \tag{1.1}$$

The Sobolev space H^s has been generalized to $B^{p,k}$ by replacing $(1 + |\xi|^2)^s$ in (1.1) by a more general tempered weight function $k(\xi)$

Received December 23, 1999.
1991 Mathematics Subject Classification: Primary 46F5, 46F12, 42B10.
Key words and phrases: Generalized Sobolev space, Bessel potential.

in Hörmander [H]. He develops a very similar theory parallel to the Sobolev spaces. Here, a positive function k defined in $\mathbb{R}^n$ is called a tempered weight function if there exist constants C and N such that $k(\xi+\eta) \le (1+C|\xi|)^n k(\eta),\ \xi,\, \eta \in \mathbb{R}^n$.

Also, the Sobolev space H^s has been generalized to H^s_w by Park and Kang [PH] by using the weight function w instead of $(1+|\xi|^2)^s$ in (1.1) satisfying the following conditions

(α) $0 = \omega(0) \le \omega(\xi+\eta) \le \omega(\xi) + \omega(\eta), \quad (\xi, \eta \in \mathbb{R}^n)$,
(β) $\int \omega(\xi)/(1+|\xi|)^{n+1}\, d\xi < \infty$,
(γ) $\omega(\xi) \ge a + b\log(1+|\xi|),\ \xi \in \mathbb{R}^n$ for some real a and positive b,
(δ) $\omega(\xi)$ is radial. i.e., $\omega(\xi) = \Omega(|\xi|)$ with Ω concave on $[0,\infty)$,

which appear in the ultradistribution theory of Beurling and Björck as in [B3].

On the other hand, R.S. Pathak [P] has introduced generalized Sobolev spaces $H^s_{\{M_k\}}$ and $H^s_{(M_k)}$ of Roumieu type and Beurling type respectively by using the L^p norms related to estimates appearing in the theory of ultradifferential functions of Roumieu and Beurling. Also, he introduced natural classes of symbols related to the theory of ultradistributions of Roumieu type, Beurling type and Beurling–Björck type and shows that its associated pseudo-differential operators act very nicely on his generalized Sobolev spaces.

In this paper we introduce generalized Sobolev spaces of exponential type by replacing $(1+|\xi|^2)^s$ in (1.1) by an exponential weight function, which is the limiting case of both generalizations by Pahk–Kang and Pathak when $w(\xi) = \xi$ in the case of Pahk–Kang and the associated function, for the defining sequence M_p, $M(\xi) = \xi$ in the case of Pathak. More clearly, the generalized Sobolev space of exponential type $H^s_{\mathcal{G}}$ consists of all $u \in \mathcal{G}'$ such that $\hat{u}$, the Fourier transform of u, is a function and satisfies

$$||u||_s = \left[\int e^{2s|\xi|}|\hat{u}(\xi)|^2 d\xi\right]^{1/2} < \infty.$$

Also, we investigate their properties such as the imbedding theorem and the structure theorem in Section 2. In fact, the imbedding theorem means that for $s > 0$ $u \in H^s_{\mathcal{G}}$ can be analytically continued to the set $\{z \in \mathbb{C}^n |\, |\mathrm{Im}\, z| < s\}$ and the structure theorem means that for $s > 0$

$u \in H_{\mathcal{G}}^{-s}$ is of the form

$$u = \sum_{\alpha} \frac{s^{|\alpha|}}{\alpha!} D^{\alpha} g_{\alpha}$$

where g_α's are square integrable functions for $\alpha \in \mathbb{N}_0^n$.

In section 3 we introduce certain classes of symbols of exponential type whose symbols have suitable growth condition and its associated pseudo-differential operators of exponential type. We show that these pseudo-dif- ferential operators naturally act on the generalized Sobolev spaces of exponential type. In other words, for r, $l \in \mathbb{R}$ with $l > 0$ the space $S_{\exp}^{r,l}$ of symbols is defined so that its associated pseudo-differential operators map $H_{\mathcal{G}}^{s}$ to $H_{\mathcal{G}}^{r+s}$. As an example, we give a differential operator of infinite order with variable coefficients of this class.

Finally, in section 4 a generalized Bessel potential is defined and its properties are investigated. The theory developed in this paper can be applied to the study of differential operators of infinite order with variable coefficients.

2. Generalized Sobolev spaces of exponential type

We first briefly introduce the space $\mathcal{G}$of test functions for Fourier ultrahyperfunctions which we need in this paper, which is introduced by di Silva.

Definition 2.1. We denote by $\mathcal{G}$ the set of all $\phi \in C^\infty(\mathbb{R}^n)$ such that for any $h, k > 0$

$$|\phi|_{h,k} = \sup_{\substack{x \in \mathbb{R}^n \\ \alpha \in \mathbb{N}_0^n}} \frac{|\partial^\alpha \phi(x)| \exp k|x|}{h^\alpha \alpha!} < \infty$$

where $\mathbb{N}_0$ is the set of nonnegative integers. The topology in $\mathcal{G}$ is defined by the above seminorms. And we denoted by $\mathcal{G}'$ the strong dual of the space $\mathcal{G}$ and call its elements Fourier ultrahyperfunctions.

Theorem 2.2 ([CK]). *The Fourier transformation on $\mathcal{G}$ is a topological isomorphism. Also, the Fourier transform on $\mathcal{G}'$ is a topological isomorphism.*

If we substitute the weight function $w(\xi)$ in the distribution space of Beurling type to $|\xi|$ (although we cannot do this), the space $\mathcal{G}'$ is just one of the Beurling's generalized distribution space. Thus it is natural to define the generalized Sobolev space of exponential type as follows:

Definition 2.3. For $s \in \mathbb{R}$, we denote by $H^s_{\mathcal{G}}$ the set of all generalized Fourier ultrahyperfunction $u \in \mathcal{G}'$ such that $\hat{u}$ is a function and

$$||u||_s = \left[\int e^{2s|\xi|}|\hat{u}(\xi)|^2 d\xi\right]^{1/2} < \infty.$$

We call $H^s_{\mathcal{G}}$ the generalized Sobolev space of exponential type of order s.

From the above definition we can easily see that $\mathcal{G}$ is contained in $H^s_{\mathcal{G}}$ for all $s \in \mathbb{R}$; $H^0_{\mathcal{G}} = L^2(\mathbb{R}^n)$. Also, we have $H^s \subset H^s_{\mathcal{G}}$ for $s < 0$ and $H^s \supset H^s_{\mathcal{G}}$ for $s > 0$. Immediately it is easy to see that $H^s_{\mathcal{G}}$ has a Hilbert space structure.

Theorem 2.4. *$H^s_{\mathcal{G}}$ is a Hilbert space with inner product given by*

$$(u, v)_s = \int e^{2s|\xi|}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi. \tag{2.1}$$

Proof. It is clearly an inner product. Since $L^2(\mathbb{R}^n, e^{2|\xi|}d\xi)$ is complete and the Fourier transform is an automorphism of $\mathcal{G}'$, $H^s_{\mathcal{G}}$ is a Banach space. □

Remark. Unfortunately $\mathcal{G}$ is not dense in $H^s_{\mathcal{G}}$.

The inclusion map and differential operators with constant coefficients are continuous.

Corollary 2.5. *$H^t_{\mathcal{G}} \subset H^s_{\mathcal{G}}$ for $t > s$, the inclusion is continuous.*

Corollary 2.6. *If P is a linear partial differential operator with constant coefficients, and $u \in H^s_{\mathcal{G}}$, then $Pu \in H^t_{\mathcal{G}}$ for all $t < s$, and the map $P : H^s_{\mathcal{G}} \to H^t_{\mathcal{G}}$ is continuous.*

REMARK. Every differential operator P with constant coefficients as in Corollary 2.6 maps H^s to H^{s-m} if the degree of the polynomial $P(\xi)$ is m.

COROLLARY 2.7. *Let $P(D) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha D^\alpha$ be a differential operator of infinite order such that there are constants $C > 0$ and $h > 0$ satisfying*

$$|a_\alpha| \le C(h/\sqrt{n})^{|\alpha|}/\alpha! \quad \text{for all } \alpha. \tag{2.2}$$

If $u \in H^s_{\mathcal{G}}$ then $Pu \in H^{s-h}_{\mathcal{G}}$. Also, the map $P : H^s_{\mathcal{G}} \to H^{s-h}_{\mathcal{G}}$ is continuous.

Proof. It is easily shown that the necessary and sufficient condition of (2.2) is that

$$|P(\xi)| \le C\exp(h|\xi|)$$

for all ξ in some tubular neighborhood of $\mathbb{R}^n$ of $\mathbb{C}^n$. Hence the result is immediate. □

As an example we give a differential operator of infinite order between generalized Sobolev spaces of exponential type, which is an isomorphism.

EXAMPLE. The operator $\exp(\sqrt{1-\Delta}) : H^s_{\mathcal{G}} \to H^{s-1}_{\mathcal{G}}$ is an isomorphism where $\exp(\sqrt{1-\Delta})$ is defined by

$$\exp(\sqrt{1-\Delta})u = \int e^{ix\xi}\exp(\sqrt{1+|\xi|^2})|\hat{u}(\xi)|d\xi.$$

Proof. For $u \in H^s_{\mathcal{G}}$ we have

$$\begin{aligned} ||e^{\sqrt{1-\Delta}}u||_{s-1} &= \left(\int e^{2(s-1)|\xi|+2\sqrt{1+|\xi|^2}}|\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\ &\le (\sup_\xi e^{2\sqrt{1+|\xi|^2}-2|\xi|})^{\frac{1}{2}}\left(\int e^{2s|\xi|}|\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\ &= e^{\frac{1}{2}}||u||_s. \end{aligned}$$

Also, for $u \in H_{\mathcal{G}}^{s-1}$ the operator $\exp(\sqrt{1-\Delta})$ maps the inverse Fourier transform of $\exp(-\sqrt{1+|\xi|^2})\hat{u}$ to u. Thus we obtain

$$\begin{aligned}&\left(\int e^{2s|\xi|-2\sqrt{1+|\xi|^2}}|\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}}\\ &\le (\sup_{\xi} e^{2|\xi|-2\sqrt{1+|\xi|^2}})^{\frac{1}{2}}\left(\int e^{2(s-1)|\xi|}|\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}}\\ &= ||u||_{s-1}.\end{aligned}$$

Hence the operator is an isomorphism. □

To find the relation between $H_{\mathcal{G}}^s$ and $H_{\mathcal{G}}^{-s}$, we define the pairing

$$\langle u, v\rangle = \int \hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi, \tag{2.3}$$

for $u \in H_{\mathcal{G}}^s$ and $v \in H_{\mathcal{G}}^{-s}$. Then we can easily obtain the following.

$$\begin{aligned}|\langle u, v\rangle| &\le \left(\int e^{2s|\xi|}|\hat{u}(\xi)|^2 d\xi\right)^{1/2}\left(\int e^{-2s|\xi|}|\bar{\hat{v}}(\xi)|^2 d\xi\right)^{1/2}\\ &= ||u||_s||v||_{-s}.\end{aligned}$$

Thus $\langle u, v\rangle$ is a continuous bilinear form on $H_{\mathcal{G}}^s \times H_{\mathcal{G}}^{-s}$.

Theorem 2.8. *The pairing* (2.3) *gives a canonical isometric isomorphism of $H_{\mathcal{G}}^{-s}$ and $(H_{\mathcal{G}}^s)'$, which is the dual of $H_{\mathcal{G}}^s$.*

Proof. It follows from (2.3) that for fixed $u \in H_{\mathcal{G}}^{-s}$, $v \mapsto \langle u, v\rangle$ is continuous linear form on $H_{\mathcal{G}}^s$, whose norm does not exceed $||u||_{-s}$. Taking $v_0 = (e^{-2s|\xi|}\hat{u}(\xi))^{\check{}} \in H_{\mathcal{G}}^s$ one can obtain that $\langle u, v_0\rangle = ||u||_{-s}$. Hence the norm of $v \to \langle u, v\rangle$ is equal to $||u||_{-s}$, and we thus have an isometry $H_{\mathcal{G}}^{-s} \to (H_{\mathcal{G}}^s)'$.

To prove that this isometry is surjective hence an isomorphism, let $u^* \in (H_{\mathcal{G}}^s)'$. By the Riesz representation theorem and (2.1) there exists $w \in H_{\mathcal{G}}^s$ such that

$$u^*(v) = (w, v)_s = \int e^{2s|\xi|}\hat{w}(\xi)\bar{\hat{v}}(\xi)d\xi \quad \text{for all} \quad v \in H_{\mathcal{G}}^s.$$

If we set $u = ((2\pi)^n \bar{w}(\xi) e^{2s|\xi|})\hat{}$ then $u \in H_{\mathcal{G}}^{-s}$ and $u^*(v) = \langle u, v\rangle$ for all $v \in H_{\mathcal{G}}^s$, which completes the proof. □

We are now in a position to state and prove the imbedding theorem and the structure theorem.

THEOREM 2.9. *Every $u \in H_{\mathcal{G}}^s$ is a holomorphic function in an infinite strip $\{z \in \mathbb{C}^n \,|\, |\mathrm{Im} z| < s\}$ if $s > 0$.*

Proof. Let $u(z) = (2\pi)^{-n} \int e^{iz\cdot\xi} \hat{u}(\xi) d\xi$ with $z = x + iy$. Then for each $\alpha \in \mathbb{N}_0^n$ we have

$$\begin{aligned}(2\pi)^{-n} \int |\xi^\alpha| e^{-y\xi - s|\xi|} |e^{s|\xi|} \hat{u}(\xi)| d\xi \\ \leq ||u||_s \left(\int |\xi^\alpha| e^{-2y\xi - 2s|\xi|} d\xi \right)^{1/2}.\end{aligned}$$

Since the integral in the last part of the above inequality is integrable if $|y| < s$, the result follows. □

THEOREM 2.10. *Let $s > 0$. Then every $u \in H_{\mathcal{G}}^{-s}$ can be represented as an infinite sum of derivatives of square integrable functions g_α, in other words,*

$$u = \sum_{\alpha \in \mathbb{N}_0^n} \frac{s^{|\alpha|}}{\alpha!} D^\alpha g_\alpha.$$

Proof. If $u \in H_{\mathcal{G}}^{-s}$ then by definition

$$\exp(-s|\xi|) \hat{u}(\xi) \in L^2(\mathbb{R}^n),$$

which implies that

$$\hat{g}(\xi) = \frac{\hat{u}(\xi)}{\sum_\alpha \frac{s^{|\alpha|}}{\alpha!} |\xi^\alpha|} \in L^2(\mathbb{R}^n).$$

Hence, we have

$$\begin{aligned}\hat{u}(\xi) &= \sum_{\alpha} \frac{s^{|\alpha|}}{\alpha!} |\xi^{\alpha}| \hat{g} \\ &= \sum_{\alpha} \frac{s^{|\alpha|}}{\alpha!} \xi^{\alpha} \left(\frac{|\xi^{\alpha}|}{\xi_{\alpha}} \hat{g} \right) \\ &= \sum_{\alpha} \frac{s^{|\alpha|}}{\alpha!} \xi^{\alpha} \hat{g}_{\alpha}\end{aligned}$$

where $\hat{g}_{\alpha}(\xi) = (|\xi^{\alpha}|/\xi^{\alpha})\hat{g}(\xi) \in L^2(\mathbb{R}^n)$. This completes the proof. □

3. Pseudo-differential operators of exponential type

The pseudo-differential operator $A(x, D)$ associated with the symbol $a(x,\xi)$ is defined by

$$(3.1) \qquad (A(x, D)u)(x) = (2\pi)^{-n/2} \int e^{ix\xi} a(x, \xi) \hat{u}(\xi) d\xi, \quad u \in \mathcal{G}$$

where $a(x, \xi)$ belongs to the class $S^r_{\exp}$, $r \geq 0$, defined below:

Definition 3.1. The function $a(x, \xi)$ is said to be in $S^r_{\exp}$ if and only if $a(x, \xi) \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ and for each compact set $K \subset \mathbb{R}^n$ and each $\alpha, \beta \in \mathbb{N}_0^n$, there exists a constant $C_K = C_{\alpha,\beta,K}$ such that the estimate

$$|D_{\xi}^{\alpha} D_x^{\beta} a(x, \xi)| \leq C_K \exp(r|\xi|), \quad \text{for all } (x, \xi) \in K \times \mathbb{R}^n$$

holds true.

Theorem 3.2. *If $a(x, \xi) \in S^r_{\exp}$ then the operator $A(x, D)$ in (3.1) is a well-defined mapping of $\mathcal{G}$ into $C^{\infty}(\mathbb{R}^n)$.*

Proof. For any compact set $K \subset \mathbb{R}^n$,

$$a(x, \xi) \leq C_K \exp(r|\xi|), \text{ for all } (x, \xi) \in K \times \mathbb{R}^n.$$

Also since $u \in \mathcal{G}$ we have

$$\int |e^{ix\xi} a(x,\xi)\hat{u}(\xi)| d\xi \leq C_K ||u||_\lambda \left(\int e^{-2(\lambda - r)|\xi|} d\xi \right)^{\frac{1}{2}}$$

is integrable for $\lambda > r$. This demonstrates the existence of $(A(x,D)u)(x)$ for all $x \in \mathbb{R}^n$ and also its continuity in $\mathbb{R}^n$. The result now follows by using Leibniz formula. □

Now we consider the symbol which belongs to the class $S^{r,l}_{\exp}$ defined below:

DEFINITION 3.3. Let $r, l \in \mathbb{R}$ be numbers with $l > 0$. The function $a(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ belongs to $S^{r,l}_{\exp}$ if and only if $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and for each $L > 0$, $\alpha, \beta \in \mathbb{N}_0^n$ there is positive constant $C = C_{r,l,\alpha}$ such that

$$|D^\alpha_\xi D^\beta_x a(x,\xi)| \leq C L^{|\beta|} |\beta|! \exp(r|\xi| - l|x|).$$

To obtain some deep and interesting results we need the following alternative form of $A(x,D)$.

THEOREM 3.4. *For any symbol $a(x,\xi) \in S^{r,l}_{\exp}$, the pseudo differential operator $A(x,D)$ admits of the representation:*

$$(A(x,D)u)(x) = (2\pi)^{-n} \int e^{ix\xi} \int \hat{a}(\xi - \eta, \eta)\hat{u}(\eta) d\eta d\xi$$

for all $u \in \mathcal{G}$ where all the involved integrals are absolutely convergent.

Proof. For $\tau > 0$ we have

$$\exp(\tau|y|) \leq \sum_k (\tau^k / k!) \sum_{|\beta| = k} (k!/\beta!) |y^\beta|.$$

So that

$$\begin{aligned}\exp(\tau|y|)|\hat{a}(y,\xi)| &\le \sum_k (\tau^k/k!) \sum_{|\beta|=k} (k!/\beta!)|y^\beta \hat{a}(y,\xi)| \\ &= \sum_k (\tau^k/k!) \sum_{|\beta|=k} (k!/\beta!)|(D_x^\beta a(x,\xi))\hat{}\,| \\ &= \sum_k (\tau^k/k!) \sum_{|\beta|=k} (k!/\beta!)(2\pi)^{-n/2} \int |D_x^\beta a(x,\xi)|dx \\ &\le C_r(2\pi)^{-n/2} \sum_k (\tau^k/k!) \sum_{|\beta|=k} (k!/\beta!) L^{|\beta|}(|\beta|!) \\ &\quad \exp(r|\xi|) \int \exp(-l|x|)dx \\ &\le C_r' \sum_{k=0}^{\infty} (\tau L n)^k \exp(r|\xi|) \\ &\le C \exp(r|\xi|)\end{aligned}$$

choosing $\tau < (Ln)^{-1}$ where C is a constant depending on r, l, n, L. Therefore

$$|\hat{a}(y,\xi)| \le C \exp(r|\xi| - \tau|y|), \quad \tau > 0. \tag{3.2}$$

Now since $u \in \mathcal{G}$, $|\hat{u}(\eta)| \le C' \exp(-\lambda|\eta|)$ for all $\lambda > 0$. Then

$$|\hat{a}(\xi-\eta,\eta)\hat{u}(\eta)| \le C'' \exp[-(\lambda - r)|\eta| - \tau|\xi-\eta|] \in L_1(\mathbb{R}^n)$$

for $\lambda > r$, $\tau > 0$. So that

$$\int |\hat{a}(\xi-\eta,\eta)\hat{u}(\eta)|d\eta \le C'' \int \exp[-(\lambda - r)|\eta| - \tau|\xi-\eta|]d\eta.$$

The right-hand side is a convolution of two integrable functions and hence is an integrable function on $\mathbb{R}^n$. Therefore the function $\int \hat{a}(\xi - \eta)\hat{u}(\eta)d\eta$ is in $L^1(\mathbb{R}^n)$. Applying inverse Fourier transform we get the result. □

Now we prove the fundamental result:

THEOREM 3.5. *Let $a(x,\xi) \in S^{r,l}_{\exp}$ and let $A(x,D)$ be the associated pseudo-differential operator. Then for all $u \in \mathcal{G}$ and all $s \in \mathbb{R}$*

$$||A(x,D)u||_s \le C_s ||u||_{r+s}. \tag{3.3}$$

Proof. Consider the function

$$U_s(\xi) = (2\pi)^{-n/2} e^{s|\xi|} \int \hat{a}(\xi-\eta)\hat{u}(\eta)d\eta, \quad s \in \mathbb{R}.$$

Then

$$|U_s(\xi)| \le (2\pi)^{-n/2} C_1 \int |e^{s|\xi-\eta|}\hat{a}(\xi-\eta,\eta)|\,|e^{s|\eta|}\hat{u}(\eta)|d\eta$$

Now, invoking inequality (3.2) we have

$$|U_s(\xi)| \le (2\pi)^{-n/2} C_1 \int \exp((s-\tau)|\xi-\eta|)\exp((r+s)|\eta|)|\hat{u}(\eta)|d\eta. \tag{3.4}$$

The integral of (3.4) can be considered as a convolution between $f(\xi) = \exp((s-\tau)|\xi|)$ and $g(\xi) = \exp((r+s)|\xi|)\hat{u}(\xi)$. Clearly $f, g \in L_2(\mathbb{R}^n)$ for $\tau > s$ since $\hat{u} \in \mathcal{G}$. Then $f * g \in L^2(\mathbb{R}^n)$ and

$$||f * g||_{L^2} \le ||f||_{L^2}||g||_{L^2}.$$

This proves (3.3). □

Some differential operator of infinite order with variable coefficients has its symbol in $S^{r,l}_{\exp}$:

COROLLARY 3.6. *If $P(x,D) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha(x)D^\alpha$ where $a_\alpha(x)$ satisfies that for each k and $\alpha, \beta \in \mathbb{N}_0^n$ there exist constants $l > 0$, $h > 0$ and $C = C_{k,l,\alpha} > 0$ such that*

$$\sup|D^\beta a_\alpha(x)| < C(h/\sqrt{n})^{|\alpha|}k^{|\beta|}|\beta!|\exp(-l|x|)/\alpha!, \tag{3.5}$$

then for $u \in \mathcal{G}$ and $s \in \mathbb{R}$ there exists a constant $C_s > 0$ such that

$$||P(x,D)u||_s \le C_s||u||_{h+s}.$$

Proof. The condition (3.5) implies that

$$\begin{aligned}|D^\beta D^\alpha P(x,\xi)| &= |\sum_{\gamma>\alpha} D^\beta a_\alpha(x)\gamma!\xi^{\gamma-\alpha}/(\gamma-\alpha)!| \\ &\le Ck^{|\beta|}|\beta!|\exp(-l|x|)(h/\sqrt{n})^{|\alpha|}\sum_\gamma (h/\sqrt{n})^\gamma \xi^\gamma/\gamma! \\ &\le C'k^{|\beta|}|\beta!|\exp(h|\xi|-l|x|).\end{aligned}$$

This completes the proof. □

4. Generalized Bessel potential of exponential type

Let $a(x) \neq 0$ be a multiplier in $\mathcal{G}$ such that $\mathfrak{F}^{-1}(a^{-m})(\xi) \in L^1(\mathbb{R}^n)$ for $m = 0, 1, \ldots$ where $\mathfrak{F}^{-1}$ is the Fourier inverse transform. Then for $f \in \mathcal{G}'$, we can define the products $fa^m \in \mathcal{G}'$ by means of the following two relation

$$\langle fa^m, \phi\rangle = \langle f, a^m\phi\rangle, \quad \phi \in \mathcal{G}$$

and

$$\langle fa^{-m}, a^m\phi\rangle = \langle f, \phi\rangle, \quad \phi \in \mathcal{G}$$

respectively, where m is a nonnegative integer. Therefore, $fa^m \in \mathcal{G}'$ for all $m \in \mathbb{Z}$.

Since the Fourier transform $\mathfrak{F}$ is a continuous linear map of $\mathcal{G}'$ onto $\mathcal{G}'$, the same being true for $\mathfrak{F}^{-1}$ also, we conclude that for $m \in \mathbb{Z}$, the generalized Bessel potential J_m defined by

$$J_m = \mathfrak{F}^{-1}a^{-m}\mathfrak{F}u, \quad u \in \mathcal{G}'$$

is a continuous linear map of $\mathcal{G}'$ onto $\mathcal{G}'$. Clearly J_m is a pseudo-differential operator with symbol a^{-m}.

The following properties of J_m can be easily be established.

Lemma 4.1. *Let $u \in \mathcal{G}'$. Then for $m, l \in \mathbb{Z}$*

(i) $J_m J_l u = J_{m+l} u$
(ii) $J_0 u = u$

For $m \in \mathbb{Z}$ and $1 \leq p \leq \infty$, define $H_{\mathcal{G}}^{m,p}$ to be the set of all Fourier ultrahyperfunctions u for which $J_m u \in L^p(\mathbb{R}^n)$. We equip this space with the norm

$$||u||_{m,p} = ||J_{-m}u||_{L^p}, \quad u \in H_{\mathcal{G}}^{m,p}. \tag{7.1}$$

THEOREM 4.2. *$H_{\mathcal{G}}^{m,p}$ is a Banach space with respect to the norm (4.1).*

THEOREM 4.3. *J_l is an isometry of $H_{\mathcal{G}}^{m,p}$ onto $H_{\mathcal{G}}^{m+l,p}$ and we have*

$$||J_l u||_{m+l,p} = ||u||_{m,p}, \quad u \in H_{\mathcal{G}}^{m,p}.$$

An analog of Sobolev imbedding theorem is the following:

THEOREM 4.4. *Let $1 < p < \infty$ and $m \leq l$. Then $H_{\mathcal{G}}^{l,p} \subset H_{\mathcal{G}}^{m,p}$, and*

$$||u||_{m,p} \leq C_{m,l}||u||_{l,p}, \quad u \in H_{\mathcal{G}}^{l,p}.$$

References

[Ba] J. Barros-Neto, *An introduction to the theory of distribution*, Marcel Dekker, INC, New York, 1973.

[Be] A. Beurling, *Quasi-analyticity and general distributions, Lectures 4 and 5*, AMS summer institute, Stanford, 1961.

[Bj] G. Björck, *Linear partial differential operators and generalized distributions*, Ark. Mat. **6** (1965), 351–407.

[CCK1] J. Chung, S.-Y. Chung and D. Kim, *Une charactérisation de l'espace de Schwartz*, C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), 23-35.

[CCK2] ———, *A characterization for the Fourier hyperfunctions*, Publ. RIMS, Kyoto Univ. **30** (1994), 203–208.

[CCK3] ———, *Characterizations of the Gelfand-Shilov Spaces via Fourier transforms*, Proc. Amer. Math. Soc. **124** (1996), 2101–2108.

[CKK] S.-Y. Chung, D. Kim, and S. K. Kim, *Structure of the extended Fourier hyperfunctions*, Japan. J. Math. **19** (1993), 1–10.

[H] L. Hörmander, *The analysis of linear partial differential operator II,*, Springer-Verlag, Berlin-New York 1983.

[P] R. S. Pathak, *Generalized Sobolev spaces and pseudo-differential operators on spaces of ultradistributions*, Structure of solutions of differential equations, Edited by M. Morimoto and T. Kawai, World Scientific, Singapore, 1996, pp. 343–368.

[PH] D. H. Pahk and B. H. Kang, *Sobolev spaces in the generalized distribution spaces of Beurling type*, Tsukuba J. Math. **15** (1991), 325–334.

Department of Mathematics
Daejin University
Pocheon, 487-800, Korea