ASSOCIATED PRIME IDEALS OF A PRINCIPAL IDEAL

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Abstract. Let $R$ be an integral domain with identity. We show that each associated prime ideal of a principal ideal in $R[X]$ has height one if and only if each associated prime ideal of a principal ideal in $R$ has height one and $R$ is an $S$-domain.

Krull’s principal ideal theorem [7, Theorem 142] states that for a nonunit element $x$ of a Noetherian ring $R$, if $P$ is a prime ideal of $R$ which is minimal over $xR$, then the height of $P$ is at most one. Thus if $R$ is a Noetherian domain then each minimal prime ideal of a nonzero principal ideal has height one. In [2], Barucci-Anderson-Dobbs studied integral domains in which each prime ideal over a nonzero principal ideal has height one. As [2], we say that an integral domain $R$ satisfies the principal ideal theorem (PIT) if each prime ideal over a nonzero principal ideal of $R$ has height one.

Let $R$ be an integral domain with identity. A prime ideal $P$ of $R$ is called an associated prime ideal of a principal ideal in $R$ if there exist some elements $a, b \in R$ such that $P$ is minimal over $aR + bR = \{x \in R | xb \in aR\}$. Consider an integral domain $R$ with the following property:

**APIT:** each associated prime ideal of a principal ideal in $R$ has height one.

One can easily show that $R$ satisfies APIT if and only if $R = \bigcap_{P \in X^1(R)} R_P$ where $X^1(R)$ is the set of all height one prime ideals of $R$ (cf. [6, Ex. 22, p.52]). The purpose of this paper is to show that $R[X]$ satisfies APIT if and only if $R$ satisfies APIT and $R$ is an $S$-domain. (Recall that an integral domain $R$ is an $S$-domain if for each height one prime ideal $P$ of $R$, the expansion $P[X]$ of $P$ to $R[X]$ has again height one.) All rings considered in this paper are commutative integral domains with identity.

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If \( a \in R \), then \( aR = aR : R \) and so a minimal prime ideal of a nonzero principal ideal is an associated prime ideal of a principal ideal. Hence \( R \) satisfies APIT then \( R \) satisfies PIT. The following example shows that the converse does not hold.

**Example 1.** Let \( R \) be the field of real numbers and let \( R[[x, y]] = R + M \) be the power series ring over \( R \), where \( M = (x, y)R[[x, y]] \). Let \( \overline{Q} \) be the algebraic closure of the field \( Q \) of rational numbers in \( R \). Let \( D := \overline{Q} + M \), then (1) \( D \) is a 2-dimensional quasi-local Mori domain with maximal \( M \), (2) \( M \) is an associated prime ideal of a principal ideal in \( D \), and hence \( D \) does not satisfies APIT, and (3) \( D \) satisfies PIT. (see [4, Example 8]).

In [5, Theorem 4], Chang proved that if \( R \) is integrally closed, then \( R[X] \) satisfies PIT if and only if \( R \) satisfies PIT and \( R \) is an \( S \)-domain. The following theorem is an APIT-analog of that fact.

**Theorem 2.** \( R[X] \) satisfies APIT if and only if \( R \) satisfies APIT and \( R \) is an \( S \)-domain.

**Proof.** (\( \Rightarrow \)) Since \( R[X] \) satisfies APIT, \( R[X] \) satisfies PIT, and \( R \) is an \( S \)-domain [2, Proposition 6.1]. Let \( P \) be an associated prime ideal of a principal ideal in \( R \), i.e., \( P \) is minimal over \( aR : bR \) for some \( a, b \in R \). Then \( P[X] \) is minimal over \( (aR : bR)R[X] \). Since \( (aR : bR)R[X] = aR[X] : bR[X] \), \( P[X] \) is an associated prime ideal of a principal ideal in \( R[X] \). Hence \( \text{ht}(P[X]) = 1 \), and so \( \text{ht}P = 1 \).

(\( \Leftarrow \)) Let \( Q \) be an associated prime ideal of a principal ideal in \( R[X] \). If \( Q \cap R = 0 \), then \( \text{ht}Q = 1 \) [7, Theorem 36]. If \( Q \cap R(= P) \neq 0 \), then \( Q = P[X] \) and \( P \) is an associated prime ideal of a principal ideal in \( R \) [3, Corollary 8]. Hence \( \text{ht}Q = \text{ht}P = 1 \). \( \square \)

Since \( R[X] \) is an \( S \)-domain [1, Theorem 3.2], it follows directly from Theorem 2 that \( R[X_1, \ldots, X_n] \) satisfies APIT if and only if \( R[X_1] \) satisfies APIT, where \( \{X_1, \ldots, X_n\} \) is a finite set of indeterminates over \( R \). It is easy to show that for nonzero elements \( a, b \in R \), \( aR\{X_\alpha\} \cap R = aR \), \( (aR : bR)R\{X_\alpha\} = aR\{X_\alpha\} : bR\{X_\alpha\} \) and \( (aR\{X_\alpha\} : bR\{X_\alpha\}) \cap R = aR : bR \) where \( \{X_\alpha\} \) is a set of indeterminates over \( R \). Using this and the proof of [2, Proposition 6.4], we have

**Corollary 3.** \( R\{X_\alpha\} \) satisfies APIT if and only if \( R \) satisfies APIT and \( R \) is an \( S \)-domain.
Given a fractional ideal $I$ of an integral domain $R$, we define $I_v = (I^{-1})^{-1}$ and $I_t = \cup\{J_v|J$ is a finitely generated subideal of $I\}$. An ideal $A$ of $R$ is said to be divisorial (resp. $t$-ideal) if $A_v = A$ (resp. $A_t = A$). Recall that an integral domain $R$ is an H-domain if each maximal $t$-ideal $P$ of $R$ is divisorial. Examples of H-domains include discrete valuation domains, Mori domains, Krull domains and Noetherian domains. It is clear that each prime $t$-ideal of $R$ has height one, then $R$ satisfies APIT (in fact, $R$ satisfies PIT). The following theorem shows that if $R$ is an H-domain, the converse also holds.

**Theorem 4.** Let $R$ be an H-domain. Then $R$ satisfies APIT if and only if each prime $t$-ideal of $R$ is a maximal $t$-ideal.

**Proof.** Suppose that $R$ satisfies APIT. Let $A$ be the set of associated prime ideals of principal ideals in $R$. Then $R = \cap_{P \in A} R_P$ [3, Proposition 4]. Let $M$ be a maximal $t$-ideal of $R$. Since $M$ is divisorial (note that $R$ is an H-domain), $R \subset M^{-1}$. Hence $M \subseteq P$ for some $P \in A$ [8, Theorem 1]. Hence $M = P$, and each maximal $t$-ideal of $R$ is an associated prime ideal of a principal ideal. Since $R$ satisfies APIT, each prime $t$-ideal is a maximal $t$-ideal. The converse is clear. \[\square\]

**Corollary 5.** If $R$ is a Noetherian ring, then $R$ satisfies APIT if and only if each prime $t$-ideal of $R$ is a maximal $t$-ideal.

**References**

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