Kangweon-Kyungki Math. Jour. **8** (2000), No. 2, pp. 103–110

PROPERTIES OF FUZZY TOPOLOGICAL GROUPS AND SEMIGROUPS

Inheung Chon

Abstract. We characterize some basic properties of fuzzy topological groups and semigroups and show that under some conditions in a fuzzy topological group G, $x \in \overline{A}$ iff $x \in \cap AU$ for any fuzzy subset A of G and the system $\{U\}$ of all fuzzy open neighborhoods of the identity e such that $U(e) = 1$.

1. Fuzzy Topological Spaces, Fuzzy Groups, and Fuzzy Semigroups

Definition 1.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in X$, $B(x)$ is called a *membership grade* of x in B. The set $\{x \in X : B(x) > 0\}$ is called the *support* of B and is denoted by supp(B).

Definition 1.2. A fuzzy topology is a family $\mathcal{T}$ of fuzzy sets in X which satisfies the following conditions:

(1) $\emptyset, X \in \mathcal{T}$,
(2) If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,
(3) If $A_i \in \mathcal{T}$ for each $i \in I$, then $\cup_{i \in I} A_i \in \mathcal{T}$.

$\mathcal{T}$ is called a *fuzzy topology* for X, and the pair $(X, \mathcal{T})$ is called a *fuzzy topological space* and is denoted by FTS for short. Every member of $\mathcal{T}$ is called $\mathcal{T}$-open fuzzy set. A fuzzy set is $\mathcal{T}$-closed iff its complement is $\mathcal{T}$-open.

Received February 15, 2000.
1991 Mathematics Subject Classification: 54A40.
Key words and phrases: fuzzy topological group.
This paper was supported by the Natural Science Research Institute of Seoul Women's University, 1999

Definition 1.3. Let A be a fuzzy set in a FTS $(X, \mathcal{T})$. The *closure* of A, denoted by $\overline{A}$, is the intersection of all closed fuzzy sets containing A. That is,

$$\overline{A} = \cap\{F : A \subseteq F \text{ and } F^c \in \mathcal{T}\}.$$

By definition ([9]), $x \in A$ iff $A(x) \neq 0$. The symbol $\emptyset$ will be used to denote an empty set, that is, $\emptyset(x) = 0$ for all $x \in X$. For X, we have by definition, $X(x) = 1$ for all $x \in X$.

Definition 1.4. Let f be a mapping from a set X to a set Y. Let A be a fuzzy set in X. Then the *image* of A, written $f(A)$, is the fuzzy set in Y with membership function defined by

$$f(A)(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} A(z), & \text{if } f^{-1}(y) \text{ is nonempty} \\ 0, & \text{otherwise,} \end{cases}$$

for all $y \in Y$. Let B be a fuzzy set in Y. Then *the inverse image* of B, written by $f^{-1}(B)$, is the fuzzy set in X with membership function defined by

$$f^{-1}(B)(x) = B(f(x)) \text{ for all } x \in X.$$

Definition 1.5. Let $(A, \mathcal{T}_A)$, $(B, \mathcal{U}_B)$ be fuzzy subspaces of FTS's $(X, \mathcal{T})$, $(Y, \mathcal{U})$, respectively. Then a map $f : (A, \mathcal{T}_A) \to (B, \mathcal{U}_B)$ is *relatively fuzzy continuous* iff for each open fuzzy set $V \in \mathcal{U}_B$, $f^{-1}(V) \cap A \in \mathcal{T}_A$. $f : (A, \mathcal{T}_A) \to (B, \mathcal{U}_B)$ is *relatively fuzzy open* iff for each open fuzzy set $W \in \mathcal{T}_A$, $f(W) \in \mathcal{U}_B$. A bijective map $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is a *fuzzy homeomorphism* iff it is fuzzy continuous and fuzzy open. A bijective map $f : (A, \mathcal{T}_A) \to (B, \mathcal{U}_B)$ is *relatively fuzzy homeomorphism* iff $f(A) = B$ and f is relatively fuzzy continuous and relatively fuzzy open.

Definition 1.6. Let X be a group and let A and B be fuzzy subsets of X. A fuzzy set A is called a *fuzzy group* in X if $A(xy) \geq \min(A(x), A(y))$ for all $x, y \in X$ and $A(x^{-1}) \geq A(x)$ for all $x \in X$. A fuzzy set B is called a *fuzzy semigroup* in X if $B(xy) \geq \min(B(x), B(y))$ for all $x, y \in X$.

It is easy to see that if A is a fuzzy group in a group X and e is the identity of X, then $A(e) \geq A(x)$ for all $x \in X$.

Proposition 1.7. *If A is a fuzzy group in a group G, then $xA = A$ if $A(x) = 1$. If S is a fuzzy subset in a group G, then for every $x, y, g \in G$,*

(1) $(xS)(g) = S(x^{-1}g)$
(2) $(Sx)(g) = S(gx^{-1})$
(3) $(xy)S = x(yS)$
(4) $S(xy) = (Sx)(y)$.

Proof. $xA(z) = \sup_{z=w_1w_2} \min(x(w_1), A(w_2)) = \min(x(x), A(x^{-1}z)) = A(x^{-1}z) \geq \min(A(x^{-1}), A(z)) = A(z)$ for all $z \in G$. Also $A(z) = A(xx^{-1}z) \geq \min(A(x), A(x^{-1}z)) = A(x^{-1}z) = xA(z)$ for all $z \in G$. The remaining thing of the proof is straightforward. □

2. Fuzzy Topological Groups and Semigroups

The following definition is due to Warren ([10]).

Definition 2.1. Let $(X, \mathcal{T})$ be a FTS. A fuzzy set N in $(X, \mathcal{T})$ is a *neighborhood* of a point $x \in X$ iff there exists $U \in \mathcal{T}$ such that $U \subseteq N$ and $U(x) = N(x) > 0$.

Definition 2.2. Let $(X, \mathcal{T})$ be a fuzzy topological space. A family $\mathcal{A}$ of fuzzy sets is a cover of a fuzzy set B iff $B \subseteq \cup_{A \in \mathcal{A}} A$. It is an open cover iff each member of $\mathcal{A}$ is an open fuzzy set. A fuzzy subset V is *fuzzy compact* iff every open cover has a finite subcover.

It is easy to see from Definition 2.2 that the continuous image of fuzzy compact set is fuzzy compact (see [2]).

Definition 2.3. Let X be a group and $\mathcal{T}$ a fuzzy topology on X. Let U, V be two fuzzy sets in X. We define UV and V^{-1} by the respective formula $UV(x) = \sup_{x=x_1x_2} \min(U(x_1), V(x_2))$ and $V^{-1}(x) = V(x^{-1})$ for $x \in X$. Let G be a fuzzy group in X and let G be endowed with the induced fuzzy topology $\mathcal{T}_G$. Then G is a *fuzzy topological group* in X, denoted by FTG for short, iff the map $\alpha : (G, \mathcal{T}_G) \times (G, \mathcal{T}_G) \to (G, \mathcal{T}_G)$ defined by $\alpha(x, y) = xy$ is relatively fuzzy continuous and the map $\beta : (G, \mathcal{T}_G) \to (G, \mathcal{T}_G)$ defined by $\beta(x) = x^{-1}$ is relatively fuzzy continuous. Let S be a fuzzy semigroup in X with induced topology $\mathcal{T}_S$. Then S is a *fuzzy topological semigroup* in X iff the map $\phi : (S, \mathcal{T}_S) \times (S, \mathcal{T}_S) \to (S, \mathcal{T}_S)$ defined by $\phi(x, y) = xy$ is relatively fuzzy continuous in both variables together.

Proposition 2.4. *Let A and B be fuzzy subsets of a fuzzy topological semigroup S in a group X and let C be fuzzy subset of a fuzzy topological group G in X.*

(1) *If A and B are fuzzy compact, then AB is fuzzy compact.*
(2) *If C is fuzzy compact, then C^{-1} is fuzzy compact.*

Proof. (1) Let $\phi : S \times S \to S$ be a map defined by $\phi(x, y) = xy$. Then ϕ is fuzzy continuous. By Theorem 3.4 in [11], $A \times B$ is compact. Since the fuzzy continuous image of fuzzy compact set is fuzzy compact, $\phi(A, B) = \phi(A \times B) = AB$ is fuzzy compact.
(2) Let $\phi : S \to S$ be a map defined by $\phi(x) = x^{-1}$. Since ϕ is fuzzy continuous and the fuzzy continuous image of fuzzy compact set is fuzzy compact, $\phi(C) = C^{-1}$ is fuzzy compact. □

The following definition is due to Warren ([10]).

Definition 2.5. A point $x \in X$ is called a *fuzzy limit point* of A iff whenever $A(x) = 1$, for each neighborhood U of x, there exists $y \in X - \{x\}$ such that $(U \cap A)(y) \neq 0$; or whenever $A(x) \neq 1$, for each open neighborhood U of x satisfying $1 - U(x) = A(x)$, there exists $y \in X - \{x\}$ such that $(U \cap A)(y) \neq 0$. A *derived fuzzy set* of A, denoted by A', is defined by

$$A'(x) = \begin{cases} \overline{A}(x), & \text{if } x \text{ is a fuzzy limit point of } A \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.6. *Let A and B be fuzzy subsets of X. If (x, y) is not a fuzzy limit point of $A \times B$, then $(\overline{A \times B})(x, y) \leq (\overline{A} \times \overline{B})(x, y)$.*

Proof. If $\overline{A \times B}(x, y) = 0$, then $\overline{A \times B}(x, y) \leq (\overline{A} \times \overline{B})(x, y)$. If $\overline{A \times B}(x, y) > 0$ and $(A \times B)(x, y) = 1$, then $\overline{A \times B}(x, y) \leq (\overline{A} \times \overline{B})(x, y)$. Suppose that $\overline{A \times B}(x, y) > 0$ and $(A \times B)(x, y) \neq 1$. Since (x, y) is not fuzzy limit point of $A \times B$, there exists open fuzzy set N, by Definition 2.5, such that $1 - N(x, y) = (A \times B)(x, y)$ and if $(c, d) \in X \times X - \{(x, y)\}$, $\min(N(c, d), (A \times B)(c, d)) = 0$. Thus for

$(c,d) \neq (x,y)$, $1 - N(c,d) \geq (A \times B)(c,d)$. Since $1 - N$ is closed and $1 - N \geq A \times B$ on $X \times X$, $1 - N \geq \overline{A \times B}$ on $X \times X$. Thus $\overline{A \times B}(x,y) \leq 1 - N(x,y) = (A \times B)(x,y) \leq (\overline{A} \times \overline{B})(x,y)$. □

Foster ([3]) showed that if G is a fuzzy topological group in a group X, the right translation $r_a : G \to G$ defined by $r_a(x) = xa$ and the left translation $l_a : G \to G$ defined by $l_a(x) = ax$ are fuzzy homeomorphisms. We review their proof and extend their results in Lemma 2.7.

LEMMA 2.7. *Let X be a group and $\mathcal{T}$ a fuzzy topology on X. Let G be a fuzzy topological group in X. Then the inversion map $f : G \to G$ defined by $f(x) = x^{-1}$ and the inner automorphism $h : G \to G$ defined by $h(g) = aga^{-1}$ are all relative fuzzy homeomorphisms, where $a \in \{x : G(x) = G(e)\}$.*

Proof. Clearly f is one-to-one. Since

$$f(G)(y) = \sup_{z \in f^{-1}(y)} G(z) = G(y^{-1}) = G(y)$$

for all $y \in G$, $f(G) = G$. Since $f^{-1}(x) = x^{-1}$ is relatively fuzzy continuous, f is relatively fuzzy open. Thus f is a relative fuzzy homeomorphism. Let $r_a : G \to G$ be a right translation defined by $r_a(x) = xa$ and $l_a : G \to G$ be a left translation defined by $l_a(x) = ax$. Then

$$\begin{aligned}(r_a(G))(x) &= \sup_{z \in r_a^{-1}(x)} G(z) = G(xa^{-1}) \\ &\geq \min(G(x), G(a^{-1})) = \min(G(x), G(e)) = G(x) = G(xa^{-1}a) \\ &\geq \min(G(xa^{-1}), G(a)) = G(xa^{-1}) = (r_a(G))(x).\end{aligned}$$

Thus $r_a(G) = G$. Let $\phi : G \to G \times G$ be a map defined by $\phi(x) = (x, a)$ and $\psi : G \times G \to G$ be a map defined by $\psi(x,y) = xy$. Then $r_a = \psi \circ \phi$. Since ϕ and ψ are fuzzy continuous, r_a is fuzzy continuous. Since $r_a^{-1} = r_{a^{-1}}$, r_a is a fuzzy homeomorphism. Similarly l_a is a fuzzy homeomorphism. Since h is a composition of $r_{a^{-1}}$ and l_a, h is a relative fuzzy homeomorphism. □

Corollary 2.8. *Let F be a fuzzy closed subset, U an fuzzy open subset, and A any fuzzy subset of a FTG G. Suppose $a \in \{x : G(x) = G(e)\}$. Then aU, Ua, U^{-1}, AU, UA are relatively open and aF, Fa, F^{-1} are relatively closed.*

Proof. Let $f : G \to G$ be a map defined by $f(x) = ax$. Since f is a relative homeomorphism, $f(U) = aU$ is relatively open. Similarly we may prove the remaining parts of the corollary. □

Proposition 2.9. *Let G be a FTG in a group X and e be an identity of G. If $a \in \{x : G(x) = G(e)\}$ and W is a neighborhood of e such that $W(e) = 1$, then aW is a neighborhood of a such that $aW(a) = 1$.*

Proof. Since W is a neighborhood of e such that $W(e) = 1$, there exists a fuzzy open set U such that $U \subseteq W$ and $U(e) = W(e) = 1$. Let $l_a : G \to G$ be a left translation defined by $l_a(g) = ag$. By Lemma 2.7, l_a is a fuzzy homeomorphism. Thus aU is a fuzzy open set. $aU(a) = U(a^{-1}a) = U(e) = 1$. $aW(x) = W(a^{-1}x) \geq U(a^{-1}x) = aU(x)$ for all $x \in X$. $aW(a) = W(a^{-1}a) = W(e) = 1$. Thus there exists an fuzzy open set aU such that $aU \subseteq aW$ and $aU(a) = aW(a) = 1$. □

Theorem 2.10. *Let G be a FTG in a group X and let $\{U\}$ be the system of all fuzzy open neighborhoods of e in a FTG G such that $U(e) = 1$, where e is the identity of X. Then for any fuzzy subset A of G, $x \in \overline{A}$ iff $x \in \cap AU$, where $x \in \{w : G(w) = G(e)\}$.*

Proof. Let $x \in \overline{A}$ and $U \in \{U\}$. Then $\overline{A}(x) > 0$. By Theorem 2.15 of [10], $\overline{A} = A \cup A'$. If $A(x) > 0$, then

$$AU(x) = \sup_{x=x_1x_2} \min(A(x_1), U(x_2)) \geq \min(A(x), U(e)) = A(x) > 0,$$

and hence $x \in AU$ for each $U \in \{U\}$, that is, $x \in \cap AU$. Suppose that $A(x) = 0$ and $A'(x) > 0$. By Theorem 2.14 of [10], x is a fuzzy limit point of A. $xU^{-1}(x) = U^{-1}(x^{-1}x) = U^{-1}(e) = U(e) = 1$. Hence $xU^{-1}(x) \geq x(x) = 1$ for all $x \in X$. Since the map $f : G \to G$ defined by $f(x) = x^{-1}$ is a fuzzy homeomorphism by Lemma 2.7, U^{-1} is fuzzy open. Since $x \in \{w : G(w) = G(e)\}$, the map $l_x : G \to G$ defined by $l_x(g) = xg$ is a fuzzy homeomorphism by Lemma 2.7. Hence

xU^{-1} is fuzzy open. $1 - xU^{-1}(x) = 1 - 1 = 0 = A(x)$. Hence xU^{-1} is a fuzzy open neighborhood of x such that $1 - xU^{-1}(x) = A(x)$. Since x is a fuzzy limit point of A, there exists $y \in X - \{x\}$ such that $(xU^{-1} \cap A)(y) \neq 0$. Since $xU^{-1}(y) = U^{-1}(x^{-1}y) = U(y^{-1}x)$, $(xU^{-1} \cap A)(y) = \min(xU^{-1}(y), A(y)) = \min(U(y^{-1}x), A(y))$. Thus $\min(U(y^{-1}x), A(y)) \neq 0$. Hence $AU(x) = \sup_{x=x_1x_2} \min(A(x_1), U(x_2)) \geq \min(A(y), U(y^{-1}x)) \neq 0$. That is, $x \in AU$ for each $U \in \{U\}$, and hence $x \in \cap AU$.

Let $x \in \cap AU$. Then $x \in AU$ for each U in $\{U\}$. If $A(x) > 0$, then $\overline{A}(x) > 0$, and hence $x \in \overline{A}$. Suppose that $A(x) = 0$. Let N be an arbitrary fuzzy open neighborhood of x such that $1 - N(x) = A(x) = 0$. Then $N(x) = 1$. $N^{-1}x(e) = N^{-1}(ex^{-1}) = N^{-1}(x^{-1}) = N(x) = 1$. By Corollary 2.8, $N^{-1}x$ is fuzzy open. Hence $N^{-1}x$ is a fuzzy open neighborhood of e. Since $N^{-1}x \in \{U\}$, $x \in AN^{-1}x$. From $AN^{-1}x(x) = AN^{-1}(xx^{-1}) = AN^{-1}(e)$ and $x \in AN^{-1}x$, $AN^{-1}(e) > 0$. Suppose that $(A \cap N)(z) = 0$ for all $z \in X - \{x\}$. Since $(A \cap N)(x) = 0$, $(A \cap N)(z) = 0$ for all $z \in X$. Then

$$\begin{aligned} AN^{-1}(e) &= \sup_{e=x_1x_2} \min(A(x_1), N^{-1}(x_2)) = \sup_x \min(A(x), N^{-1}(x^{-1})) \\ &= \sup_x \min(A(x), N(x)) = \sup_x (A \cap N)(x) = 0. \end{aligned}$$

This contradicts $AN^{-1}(e) > 0$. Thus there exists $y \in X - \{x\}$ such that $(A \cap N)(y) \neq 0$. Hence x is a fuzzy limit point of A. By Theorem 2.14 of [10], $A'(x) > 0$. Thus $\overline{A}(x) \geq A'(x) > 0$, that is, $x \in \overline{A}$. □

References

1. M. Asaad and S. Abou-Zaid, *A contribution to the theory of fuzzy subgroups*, Fuzzy Sets and Systems **77** (1996), 355–369.
2. C. L. Chang, *Fuzzy Topological Spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.
3. D. H. Foster, *Fuzzy Topological Groups*, J. Math. Anal. Appl. **67** (1979), 549–564.
4. T. E. Gantner, R. C. Steinlage, and R. H. Warren, *Compactness in Fuzzy Topology Spaces*, J. Math. Anal. Appl. **62** (1978), 547–562.
5. R. Lowen, *Fuzzy Topological Spaces and Fuzzy Compactness*, J. Math. Anal. Appl. **56** (1976), 621–633.

6. R. Lowen, *A Comparison of Different Compactness Notions in Fuzzy Topological Spaces*, J. Math. Anal. Appl. **64** (1978), 446–454.
7. A. Rosenfeld, *Fuzzy Groups*, J. Math. Anal. Appl. **35** (1971), 512–517.
8. M. W. Warner, *On fuzzy compactness*, Fuzzy Sets and Systems **80** (1996), 15–22.
9. R. H. Warren, *Optimality in fuzzy topological polysystems*, J. Math. Anal. Appl. **54** (1976), 309–315.
10. R. H. Warren, *Neighborhoods, Bases, and Continuity in Fuzzy Topological Spaces*, Rocky Mountain Journal of Mathematics **8** (1978), 459–470.
11. C. K. Wong, *Fuzzy Topology: Products and Quotient Theorems*, J. Math. Anal. Appl. **45** (1974), 512–521.

Department of Mathematics
Seoul Women's University
Seoul 139-774, Korea