ASYMPTOTIC PROPERTIES OF NONEXPANSIVE SEQUENCES IN BANACH SPACES

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Abstract. B. Djafari Rouhani and W. A. Kirk [3] proved the following theorem:
Let $X$ be a reflexive Banach space and $(x_n)_{n\geq 0}$ be a nonexpansive (resp., firmly nonexpansive) sequence in $X$. Then the set of weak $\omega$-limit points of the sequence $(\frac{x_n}{n})_{n\geq 1}$ (resp., $(x_n - x_{n+1})_{n\geq 0}$) always lies on a convex subset of a sphere centered at the origin of radius $d = \lim_{n\to\infty} \|x_n\|$.
In this paper we show that the above theorem for nonexpansive (resp., firmly nonexpansive) sequences holds in a general Banach space (resp., a strictly convex dual $X^*$).

1. Introduction

Let $X$ be a real Banach space; the norm of both $X$ and its dual $X^*$ are denoted by $\|\|$; we denote strong convergence and weak convergence in $X$ respectively by $\to$ and $\rightharpoonup$. The duality map $J$ from $X$ into the family of nonempty closed convex subsets of $X^*$ is defined by $J(x) = \{x^* \in X^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}$. We say that a sequence $(x_n)_{n\geq 0}$ is nonexpansive (resp., firmly nonexpansive) if $\|x_{i+1} - x_{j+1}\| \leq \|x_i - x_j\|$ for all $i, j \geq 0$ (resp., if the function $f : [0, 1] \to [0, \infty)$ defined by $f(t) = \|(1-t)(x_i - x_j) + t(x_{i+1} - x_{j+1})\|$ is nonincreasing for all $i, j \geq 0$).
Firmly nonexpansive sequences are also characterized by the inequality
$$\|x_{i+1} - x_{j+1}\| \leq \|(1-t)(x_i - x_j) + t(x_{i+1} - x_{j+1})\|$$
for all $i, j \geq 0$, $t \in [0, 1]$.

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In 1971 A. Pazy proved that if $T$ is nonexpansive in a Hilbert space, then sequence $\{T^n x/n\}$ always converges strongly. Since then the asymptotic behavior of nonexpansive mappings has been extended to a more general space and to the firmly nonexpansive mappings and to nonexpansive semigroups. (See, e.g., [4, 5]).

In this context B. Djafari Rouhani [1, 2] defined nonexpansive sequences and firmly nonexpansive sequences and studied their asymptotic behaviors.

The present paper concerns about Proposition 1 and Theorem 1,2 in [3] without reflexivity.

2. Asymptotic Behavior

We follow the notations in [3],

$$K_n = \overline{\text{conv}} \{(x_{i+1} - x_i)\}_{i \geq n}$$

$$K = \bigcap_{n=0}^{\infty} K_n$$

$$F_n = \overline{\text{conv}} \{(x_k - x_0)/k\}_{k \geq n}$$

$$F = \bigcap_{n=0}^{\infty} F_n$$

$$S_d = \{x \in X : \|x\| = d\}.$$ 

Remark 1 in [3] states that if $H_n := \overline{\text{conv}} \{(x_k - x_0)/k\}_{k \geq n}$ and $H = \bigcap_{n=1}^{\infty} H_n$, then $H = F$.

In analogy of Lemma 4 in [5] we can obtain the following lemma.

**Lemma 1.** Let $(x_n)_{n \geq 0}$ be a nonexpansive sequences in a Banach space $X$. And let $d = \lim_{n \to \infty} \|x_n\|$. Then there exists $x^* \in X^*$ such that

$$(x^*, x_m - x_0)/m \geq \|x^*\|^2 = d^2$$

for all $m \geq 1$.

**Proof.** The proof is similar to that of Lemma 4 in [5]. So it is omitted.
Theorem 1. Let $X$ be a Banach space, $(x_n)_{n \geq 0}$ a nonexpansive sequence in $X$, and $d = \lim_{n \to \infty} \| \frac{x_n}{n} \|$. Then
\[
\omega_w(\{ \frac{x_n}{n} \}) \subseteq F \cap S_d.
\]

Proof. Suppose that $\{ \frac{x_{nk}}{n_k} \}$ converges weakly to $z \in \omega_w(\{ \frac{x_n}{n} \})$. Then
\[
\| z \| \leq \liminf_{n_k \to \infty} \| \frac{x_{nk}}{n_k} \| = d
\]
and by Lemma 1,
\[
(x^*, z) = \lim_{n_k \to \infty} (x^*, \frac{x_{nk}}{n_k}) \geq d^2.
\]
So $\| z \| \geq d$ and $z \in S_d$.

Since $H_n$ is closed and convex, it is weakly closed and $z \in H_n$ for all $n \geq 1$. Therefore $\| z \| \in \cap_{n=1}^\infty H_n = H = F$. \qed

In [3] B. Djafari Rouhani and W.A. Kirk characterized the sets $F \cap S_d$, $K \cap S_d$. Without reflexivity of $X$, we prove the following theorem.

Theorem 2. Let $(x_n)_{n \geq 0}$ be a nonexpansive sequence in a Banach space $X$. Then $F_n \cap S_d$ is convex for all $n \geq 1$. In particular $F \cap S_d$ is convex.

Proof. It is sufficient to prove that for all $x, y \in F_n \cap S_d, 0 \leq \lambda \leq 1$,
\[
\| (1 - \lambda)x + \lambda y \| = d.
\]
By Lemma 1, for all $w \in F_n$
\[
(x^*, w) \geq d^2.
\]
So
\[
(x^*, (1 - \lambda)x + \lambda y) \geq d^2.
\]
Hence
\[
\| (1 - \lambda)x + \lambda y \| \geq d.
\]
On the other hand, since $x, y \in S_d$
\[
\| (1 - \lambda)x + \lambda y \| \leq d.
\]
Therefore
\[
\| (1 - \lambda)x + \lambda y \| = d.
\]
\qed
Being hinted by Corollary 2 of A.T. Plant and S. Reich [5], we obtain the following lemma which could be compared with Theorem 3.1 of [3]. We need Lemma 5 in [5] which characterizes the strict convexity as the duality map as follows:

A Banach space $X$ is strictly convex iff its duality map is injective in the sense that $J(x) \cap J(y) \neq \emptyset$ implies $x = y$.

**Lemma 2.** If $(x_n)_{n \geq 0}$ is a nonexpansive sequence in a Banach space $X$ with strictly convex dual $X^*$ and let $d = \lim_{n \to \infty} \|x_n\|$, then there exists $z \in X^*$ such that

$$\left( z, \frac{x_{m+i} - x_i}{m} \right) \geq \|z\|^2 = d^2$$

for all $m \geq 1, i \geq 0$.

**Proof.** Since $\{x_n\}$ is bounded in $X$ which is identified with its natural injection in $X^{**}$, let us choose one weak-star subsequential limit $a$ of $\{x_n\}$. So

$$\|a\| \leq \liminf_{n \to \infty} \frac{x_n}{n} = d$$

And for any $i \geq 0$, the subsequence $(x_{n+i})_{n \geq 0}$ is also a nonexpansive sequence. So by Lemma 1 there exists $z(i) \in X^*$ such that

$$\left( z(i), \frac{x_{m+i} - x_i}{m} \right) \geq \|z(i)\|^2 = d^2 \ldots \ldots (\star)$$

for all $m \geq 1$. Here

$$\lim_{m \to \infty} \left| \frac{x_{m+i}}{m} \right| = \lim_{m \to \infty} \left| \left( \frac{m+i}{m} \right) \frac{x_{m+i}}{m+i} \right| = \lim_{n \to \infty} \frac{x_n}{n} = d.$$ 

Since some subsequence of $\{x_{m+i} = (\frac{m+i}{m}) \frac{x_{m+i}}{m+i}\}$ converges weak-star to $a \in X^{**}$, in $(\star)$ we take the subsequential limit. Then

$$\left( z(i), a \right) \geq \|z(i)\|^2 = d^2.$$ 

And

$$d^2 \geq \|z(i)\| \|a\| \geq \left( z(i), a \right) \geq \|z(i)\|^2 = d^2.$$ 

Therefore

$$\left( z(i), a \right) = \|a\|^2 = \|z(i)\|^2.$$ 

for all $i \geq 0$. So for the usual duality map $J^*$ on $X^{**}$,

$$a \in J^*(z(i)) \cap J^*(z(j)).$$
for all \( i, j \geq 0 \). Since \( X^* \) is strictly convex, \( z \equiv z(i) \) for all \( i \geq 0 \). Hence
\[
(z, \frac{x_{m+i} - x_i}{m}) \geq \|z\|^2 = d^2
\]
for all \( m \geq 1, i \geq 0 \).

It is known in [2, 6] that for a firmly nonexpansive sequence \((x_n)_{n \geq 0}\),
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = d = \lim_{n \to \infty} \|\frac{x_n}{n}\|.
\]

**Theorem 3.** Let \( X \) be a Banach space with a strictly convex dual \( X^* \), \((x_n)_{n \geq 0}\) a firmly nonexpansive sequence in \( X \), and \( d = \lim_{n \to \infty} \|\frac{x_n}{n}\| = \lim_{n \to \infty} \|x_{n+1} - x_n\| \). Then
\[
\omega_w(\{x_{n+1} - x_n\}) \subseteq K \cap S_d.
\]

**Proof.** Since a firmly nonexpansive sequence is a nonexpansive sequence, there exists \( z \in X^* \) such that
\[
(z, x_{i+1} - x_i) \geq \|z\|^2 = d^2
\]
for all \( i \geq 0 \). by Lemma 2 for \( m = 1 \). So by the definition of \( K_n \),
\[
(z, w) \geq \|z\|^2 = d^2
\]
for all \( w \in K_n, n \geq 0 \). If \( x_{n+1} - x_n \) converges weakly to \( w' \in X \), then
\[
\|w'\| \leq \liminf_{n \to \infty} \|x_{n+1} - x_n\| = d
\]
on the other hand since \( K_n \) is weakly closed, \( w' \in K_n \) for all \( n \geq 0 \) i.e.,
\[
w' \in \cap_{n=0}^{\infty} K_n = K \quad \text{and} \quad (z, w') \geq \|z\|^2 = d^2.
\]
So \( \|w'\| \leq d \). Therefore \( w' \in K \cap S_d \).

**Theorem 4.** Let \((x_n)_{n \geq 0}\) be a firmly nonexpansive sequence in a Banach space \( X \) with a strictly convex dual \( X^* \). Then \( K \cap S_d \) is convex for all \( n \geq 1 \). In particular \( K \cap S_d \) is convex.

**Proof.** The proof depends on Lemma 2 for \( m = 1 \). And its proof is similar to that of Theorem 2. So we omit it.

**References**


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