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ON A SUBCLASS OF PRESTALIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. Motivated by recent work of Uralegaddi and Sarangi[12], we aim at presenting here system study of novel subclass $R_\alpha[\mu, \beta, \xi]$ of prestarlike functions. Further using operators of fractional calculus, we have obtained distortion theorem for $R_\alpha[\mu, \beta, \xi]$. Lastly the extreme points of $R_\alpha[\mu, \beta, \xi]$ are obtained.

1. Introduction.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in the unit disc $U = \{z : |z| < 1\}$, let S denote the subclass of A consisting of analytic and univalent functions $f(z)$ in the unit disc U. Further T denote subclass of A consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0. \tag{1.2}$$

Schild[8] studied a subclass of S consisting of polynomials having $|z| = 1$ as radius univalence. Subsequently, Silverman[10] proved useful results for the subclasses $S^*(\alpha)$ and $C(\alpha)$ of S, where $S^*(\alpha)$ and $C(\alpha)$ denote respectively, the subclasses of starlike functions of order α and convex functions of order α, $0 \leq \alpha < 1$. We note that $S^*(\alpha)$ was introduced by Robertson[5].

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The function

$$S_\alpha(z) = z(1-z)^{-2(1-\alpha)} \tag{1.3}$$

is the well known extremal function for the class $S^*(\alpha)$. Letting,

$$C(\alpha, n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!}, \qquad n = 2, 3, \cdots \tag{1.4}$$

$S_\alpha(z)$ can be written in the form

$$S_\alpha(z) = z + \sum_{n=2}^{\infty} C(\alpha, n) z^n. \tag{1.5}$$

We note that $C(\alpha, n)$ is decreasing in α and satisfies

$$\lim_{n\to\infty} C(\alpha, n) = \begin{cases} \infty & \alpha < \frac{1}{2} \\ 1 & \alpha = \frac{1}{2} \\ 0 & \alpha > \frac{1}{2}. \end{cases} \tag{1.6}$$

Let $(f * g)(z)$ denote the convolution or Hadamard product of $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.8}$$

Let $R_\alpha(\mu, \beta, \xi)$ denote the class of prestarlike functions, that satisfies the condition

$$\left| \frac{\frac{zh'(z)}{h(z)} - 1}{2\xi(\frac{zh'(z)}{h(z)} - \mu) - (\frac{zh'(z)}{h(z)} - 1)} \right| < \beta \tag{1.9}$$

where, $h(z) = f * S_\alpha(z)$, $0 < \beta \leq 1$, $0 \leq \mu < 1$, $1/2 < \xi \leq 1$. The class of α-prestarlike functions was introduced by Ruscheweyh[7] and later on rather extensively studied by Silverman and Silvia[9], Owa and Ahuja[4] and Uralegaddi and Sarangi[12].

Let

$$R_\alpha[\mu, \beta, \xi] = R_\alpha(\mu, \beta, \xi) \cap T. \tag{1.10}$$

Our main tool in the present paper is the following, which can be easily proved, the details are omitted.

Lemma 1. *Let $f(z)$ be defined by (1.2), then $f(z)$ is in the class $R_\alpha[\mu, \beta, \xi]$ if and only if*

$$\sum_{n=2}^{\infty} ((n-1) + \beta(2\xi(n-\mu) - (n-1)))\, C(\alpha, n) a_n \leq 2\beta\xi(1-\mu).$$

The result is sharp.

2. Distortion Theorems Involving Fractional Calculus

In this section, we shall prove distortion theorems for functions belonging to the class $R_\alpha[\mu, \beta, \xi]$. Each of these would involve operators of fractional calculus which are defined as follows (cf. e.g [2, 3, 6, 11]).

Defintion 1. The fractional integral of order λ is defined by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \tag{2.1}$$

where $\lambda > 0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2. The fractional derivative of order λ is defined by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta, \tag{2.2}$$

where $0 \leq \lambda < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1.

Definition 3. Under the hypothesis of Definition 2, the fractional derivative of order $n + \lambda$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \tag{2.3}$$

where $0 \leq \lambda < 1$, $n \in N \cup \{0\}$, $N = \{1, 2, \cdots\}$.

THEOREM 1. *Let $f(z)$ given by (1.2) be in the class $R_\alpha[\mu, \beta, \xi]$. Then*

$$|D_z^{-\lambda} f(z)| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left(1 - \frac{\beta\xi(1-\mu)}{(1+\beta(4\xi - 2\xi\mu - 1))(1-\alpha)} |z| \right) \tag{2.4}$$

and

$$|D_z^{-\lambda} f(z)| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left(1 + \frac{\beta\xi(1-\mu)}{(1+\beta(4\xi - 2\xi\mu - 1))(1-\alpha)} |z| \right) \tag{2.5}$$

for $\lambda > 0$, $z \in U$. The bounds are sharp.

Proof. Let

$$\begin{aligned} F(z) &= \Gamma(2+\lambda) z^{-\lambda} D_z^{-\lambda} f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n z^n \end{aligned} \tag{2.6}$$

for $\lambda > 0$. We note that

$$0 < \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} < n \tag{2.7}$$

for $\lambda > 0$, $n \geq 2$, and that $C(\alpha, n+1) \geq C(\alpha, n)$, for $0 \leq \alpha < 1/2$, and $n \geq 2$. Consequently, by using Lemma 1, we have

$$\begin{aligned} |F(z)| &\geq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n \\ &\geq |z| - \frac{\beta\xi(1-\mu)}{(1+\beta(4\xi - 2\xi\mu - 1))(1-\alpha)} |z|^2 \end{aligned}$$

which implies (2.4), and

$$\begin{aligned} |F(z)| &\leq |z| + |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n \\ &\leq |z| + \frac{\xi\beta(1-\mu)}{(1+\beta(4\xi - 2\xi\mu - 1))(1-\alpha)} |z|^2 \end{aligned}$$

which gives (2.5).

The result is sharp for the function $f(z)$ given by

$$D_z^{-\lambda} f(z) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left(1 - \frac{\beta\xi(1-\mu)}{(1+\beta(4\xi - 2\xi\mu - 1))(1-\alpha)} z \right).$$

$\square$

Corollary 1. *Let the functions $f(z)$ be defined by (1.2) is in the class $R_\alpha[\mu,\beta,\xi]$, with $0 \le \alpha \le 1/2$, $1/2 < \xi \le 1$, $0 < \beta \le 1$ and $0 \le \mu < 1$. Then $D_z^{-\lambda} f(z)$ is included in a disc with center at origin and radius r_1 given by*

$$r_1 = \frac{1}{\Gamma(2+\lambda)}\left(1 + \frac{\beta\xi(1-\mu)}{(1+\beta(4\xi-2\xi\mu-1))(1-\alpha)}\right), \tag{2.8}$$

where $\lambda > 0$.

Theorem 2. *Let the functions $f(z)$ given by (1.2) be in the class $R_\alpha[\mu,\beta,\xi]$. Then*

$$|D_z^\lambda f(z)| \ge \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left(1 - \frac{2\xi\beta(1-\mu)}{(1+\beta(4\xi-4\xi\mu-1))(1-\alpha)}|z|\right) \tag{2.9}$$

and

$$|D_z^\lambda f(z)| \le \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left(1 + \frac{2\xi\beta(1-\mu)}{(1+\beta(4\xi-4\xi\mu-1))(1-\alpha)}|z|\right) \tag{2.10}$$

for $0 \le \lambda < 1$, $z \in U$. The bounds are sharp.

Proof. Let

$$\begin{aligned} G(z) &= \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n \end{aligned}$$

for $0 \le \lambda < 1$. By using Lemma 1, we observe that

$$\begin{aligned} &\frac{1}{2}(1+\beta(4\xi-4\xi\mu-1))\, C(\alpha,2) \sum_{n=2}^{\infty} n a_n \\ &\le \sum_{n=2}^{\infty} \big((n-1)+\beta(2\xi(n-\mu)-(n-1))\big) C(\alpha,n) a_n \\ &\le 2\beta\xi(1-\mu), \end{aligned} \tag{2.11}$$

which implies that

$$\sum_{n=2}^{\infty} na_n \leq \frac{2\xi\beta(1-\mu)}{(1+\beta(4\xi-4\xi\mu-1))(1-\alpha)}. \tag{2.12}$$

Further, we note that $1 < \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} < n$ for $0 \leq \lambda < 1,\ n \geq 2$. Hence we have

$$\begin{aligned}|G(z)| &\geq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n \\ &\geq |z| - |z|^2 \sum_{n=2}^{\infty} na_n \\ &\geq |z| - \frac{2\xi\beta(1-\mu)}{(1+\beta(4\xi-4\xi\mu-1))(1-\alpha)}|z|^2\end{aligned}$$

which proves (2.9), and

$$\begin{aligned}|G(z)| &\leq |z| + |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n \\ &\leq |z| + |z|^2 \sum_{n=2}^{\infty} na_n \\ &\leq |z| + \frac{2\xi\beta(1-\mu)}{(1+\beta(4\xi-4\xi\mu-1))(1-\alpha)}|z|^2\end{aligned}$$

which gives (2.10).

Finally, The bound of (2.9) and (2.10) are sharp, extremal function being

$$D_z^{\lambda} f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \frac{2\xi\beta(1-\mu)}{(1+\beta(4\xi-4\xi\mu-1))(1-\alpha)}\right).$$

□

Corollary 2. *Let the function $f(z)$ given by (1.2) be in the class $R_\alpha[\mu,\beta,\xi]$. Then $D_z^\lambda f(z)$ is included in the disc with center at origin and radius r_2 given by,*

$$r_2 = \frac{1}{\Gamma(2-\lambda)}\left(1 + \frac{2\xi\beta(1-\mu)}{(1+\beta(4\xi-4\xi\mu-1))(1-\alpha)}\right), \tag{2.13}$$

where $0 \le \lambda < 1$.

Finally, we obtain extreme points of $R_\alpha[\mu,\beta,\xi]$ by the routine calculation.

Theorem 3. *Let*

$$f_1(z) = z$$

and

$$f_n(z) = z - \frac{2\beta\xi(1-\mu)}{\big((n-1)+\beta(2\xi(n-\mu)-(n-1))\big)C(\alpha,n)} z^n, \tag{2.14}$$

$n = 2, 3, \cdots$. *Then $f \in R_\alpha[\mu,\beta,\xi]$ if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \ge 0$, $\sum_{n=1}^{\infty} \lambda_n = 1$.

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