ON A SUBCLASS OF PRESTALIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. Motivated by recent work of Uralegaddi and Sarangi[12], we aim at presenting here system study of novel subclass $R_{\alpha}[\mu, \beta, \xi]$ of prestarlike functions. Further using operators of fractional calculus, we have obtained distortion theorem for $R_{\alpha}[\mu, \beta, \xi]$. Lastly the extreme points of $R_{\alpha}[\mu, \beta, \xi]$ are obtained.

1. Introduction.

Let $A$ denote the class of functions of the form

\begin{equation}
  f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\end{equation}

that are analytic in the unit disc $U = \{z : |z| < 1\}$, let $S$ denote the subclass of $A$ consisting of analytic and univalent functions $f(z)$ in the unit disc $U$. Further $T$ denote subclass of $A$ consisting of functions $f(z)$ of the form

\begin{equation}
  f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0.
\end{equation}

Schild[8] studied a subclass of $S$ consisting of polynomials having $|z| = 1$ as radius univalence. Subsequently, Silverman[10] proved useful results for the subclasses $S^*(\alpha)$ and $C(\alpha)$ of $S$, where $S^*(\alpha)$ and $C(\alpha)$ denote respectively, the subclasses of starlike functions of order $\alpha$ and convex functions of order $\alpha$, $0 \leq \alpha < 1$. We note that $S^*(\alpha)$ was introduced by Robertson[5].

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The function

\[ S_\alpha(z) = z(1 - z)^{-2(1-\alpha)} \]  

is the well known extremal function for the class \( S^*(\alpha) \). Letting,

\[ C(\alpha, n) = \prod_{k=2}^{n} \frac{(k - 2\alpha)}{(n - 1)!}, \quad n = 2, 3, \ldots \]

\( S_\alpha(z) \) can be written in the form

\[ S_\alpha(z) = z + \sum_{n=2}^{\infty} C(\alpha, n)z^n. \]

We note that \( C(\alpha, n) \) is decreasing in \( \alpha \) and satisfies

\[ \lim_{n \to \infty} C(\alpha, n) = \begin{cases} \infty & \alpha < \frac{1}{2} \\ 1 & \alpha = \frac{1}{2} \\ 0 & \alpha > \frac{1}{2}. \end{cases} \]

Let \((f \ast g)(z)\) denote the convolution or Hadamard product of \( f(z) \) given by (1.1) and \( g(z) \) given by

\[ g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \]

then

\[ (f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \]

Let \( R_\alpha(\mu, \beta, \xi) \) denote the class of prestarlike functions, that satisfies the condition

\[ \frac{|zh'(z)|}{h(z)} - 1 < \beta \]

where, \( h(z) = f \ast S_\alpha(z) \), \( 0 < \beta \leq 1 \), \( 0 \leq \mu < 1 \), \( 1/2 < \xi \leq 1 \). The class of \( \alpha \)-prestarlike functions was introduced by Ruscheweyh[7] and later on rather extensively studied by Silverman and Silvia[9], Owa and Ahuja[4] and Uralegaddi and Sarangi[12].

Let

\[ R_\alpha[\mu, \beta, \xi] = R_\alpha(\mu, \beta, \xi) \cap T. \]

Our main tool in the present paper is the following, which can be easily proved, the details are omitted.
Lemma 1. Let \( f(z) \) be defined by (1.2), then \( f(z) \) is in the class \( R_\alpha[\mu, \beta, \xi] \) if and only if
\[
\sum_{n=2}^{\infty} ((n-1) + \beta(2\xi(n-\mu) - (n-1))) C(\alpha, n)a_n \leq 2\beta\xi(1-\mu).
\]
The result is sharp.

2. Distortion Theorems Involving Fractional Calculus

In this section, we shall prove distortion theorems for functions belonging to the class \( R_\alpha[\mu, \beta, \xi] \). Each of these would involve operators of fractional calculus which are defined as follows (cf. e.g [2, 3, 6, 11]).

Definition 1. The fractional integral of order \( \lambda \) is defined by
\[
D_{z}^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)} \int_{0}^{z} \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta,
\]
where \( \lambda > 0 \), \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin, and the multiplicity of \((z - \zeta)^{\lambda-1}\) is removed by requiring \( \log(z - \zeta) \) to be real when \( z - \zeta > 0 \).

Definition 2. The fractional derivative of order \( \lambda \) is defined by
\[
D_{z}^{\lambda}f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_{0}^{z} \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta,
\]
where \( 0 \leq \lambda < 1 \), \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin and the multiplicity of \((z - \zeta)^{-\lambda}\) is removed as in Definition 1.

Definition 3. Under the hypothesis of Definition 2, the fractional derivative of order \( n + \lambda \) is defined by
\[
D_{z}^{n+\lambda}f(z) = \frac{d^n}{dz^n} D_{z}^{\lambda}f(z),
\]
where \( 0 \leq \lambda < 1 \), \( n \in N \cup \{0\} \), \( N = \{1, 2, \cdots \} \).
Theorem 1. Let \( f(z) \) given by (1.2) be in the class \( R_\alpha[\mu, \beta, \xi] \). Then

\[
(2.4) \quad |D_z^{-\lambda}f(z)| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left( 1 - \frac{\beta\xi(1-\mu)}{(1+\beta(4\xi-2\xi\mu-1))(1-\alpha)}|z| \right)
\]

and

\[
(2.5) \quad |D_z^{-\lambda}f(z)| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left( 1 + \frac{\beta\xi(1-\mu)}{(1+\beta(4\xi-2\xi\mu-1))(1-\alpha)}|z| \right)
\]

for \( \lambda > 0, \ z \in U \). The bounds are sharp.

Proof. Let

\[
F(z) = \Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}f(z)
\]

\[
= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n z^n
\]

for \( \lambda > 0 \). We note that

\[
0 < \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} < n
\]

for \( \lambda > 0, \ n \geq 2 \), and that \( C(\alpha, n+1) \geq C(\alpha, n) \), for \( 0 \leq \alpha < 1/2 \), and \( n \geq 2 \). Consequently, by using Lemma 1, we have

\[
|F(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n
\]

\[
\geq |z| - \frac{\beta\xi(1-\mu)}{(1+\beta(4\xi-2\xi\mu-1))(1-\alpha)}|z|^2
\]

which implies (2.4), and

\[
|F(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n
\]

\[
\leq |z| + \frac{\xi\beta(1-\mu)}{(1+\beta(4\xi-2\xi\mu-1))(1-\alpha)}|z|^2
\]

which gives (2.5).

The result is sharp for the function \( f(z) \) given by

\[
D_z^{-\lambda}f(z) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left( 1 - \frac{\beta\xi(1-\mu)}{(1+\beta(4\xi-2\xi\mu-1))(1-\alpha)}|z| \right).
\]
Corollary 1. Let the functions $f(z)$ be defined by (1.2) be in the class $R_{\alpha}[\mu, \beta, \xi]$, with $0 \leq \alpha \leq 1/2$, $1/2 < \xi \leq 1$, $0 < \beta \leq 1$ and $0 \leq \mu < 1$. Then $D_{z}^{-\lambda}f(z)$ is included in a disc with center at origin and radius $r_{1}$ given by

\begin{equation}
(2.8) \quad r_{1} = \frac{1}{\Gamma(2 + \lambda)} \left( 1 + \frac{\beta \xi(1 - \mu)}{(1 + \beta(4\xi - 2\xi\mu - 1))(1 - \alpha)} \right),
\end{equation}

where $\lambda > 0$.

Theorem 2. Let the functions $f(z)$ given by (1.2) be in the class $R_{\alpha}[\mu, \beta, \xi]$. Then

\begin{equation}
(2.9) \quad |D_{z}^{\lambda}f(z)| \geq \frac{|z|^{1 - \lambda}}{\Gamma(2 - \lambda)} \left( 1 - \frac{2\xi\beta(1 - \mu)}{(1 + \beta(4\xi - 4\xi\mu - 1))(1 - \alpha)|z|} \right)
\end{equation}

and

\begin{equation}
(2.10) \quad |D_{z}^{\lambda}f(z)| \leq \frac{|z|^{1 - \lambda}}{\Gamma(2 - \lambda)} \left( 1 + \frac{2\xi\beta(1 - \mu)}{(1 + \beta(4\xi - 4\xi\mu - 1))(1 - \alpha)|z|} \right)
\end{equation}

for $0 \leq \lambda < 1$, $z \in U$. The bounds are sharp.

Proof. Let

\begin{align*}
G(z) &= \Gamma(2 - \lambda)z^{\lambda}D_{z}^{\lambda}f(z) \\
&= z - \sum_{n=2}^{\infty} \frac{\Gamma(n + 1)\Gamma(2 - \lambda)}{\Gamma(n + 1 - \lambda)} a_{n}z^{n}
\end{align*}

for $0 \leq \lambda < 1$. By using Lemma 1, we observe that

\begin{equation}
(2.11) \quad \frac{1}{2} \left( 1 + \beta(4\xi - 4\xi\mu - 1) \right) C(\alpha, 2) \sum_{n=2}^{\infty} na_{n} \\
\leq \sum_{n=2}^{\infty} \left( (n - 1) + \beta(2\xi(n - \mu) - (n - 1)) \right) C(\alpha, n)a_{n} \\
\leq 2\beta \xi(1 - \mu),
\end{equation}
which implies that

\[
\sum_{n=2}^{\infty} na_n \leq \frac{2\xi\beta(1 - \mu)}{(1 + \beta(4\xi - 4\xi\mu - 1))(1 - \alpha)}.
\]

Further, we note that \(1 < \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} < n\) for \(0 \leq \lambda < 1, \ n \geq 2\). Hence we have

\[
|G(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n
\geq |z| - |z|^2 \sum_{n=2}^{\infty} na_n
\geq |z| - \frac{2\xi\beta(1 - \mu)}{(1 + \beta(4\xi - 4\xi\mu - 1))(1 - \alpha)}|z|^2
\]

which proves (2.9), and

\[
|G(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n
\leq |z| + |z|^2 \sum_{n=2}^{\infty} na_n
\leq |z| + \frac{2\xi\beta(1 - \mu)}{(1 + \beta(4\xi - 4\xi\mu - 1))(1 - \alpha)}|z|^2
\]

which gives (2.10).

Finally, The bound of (2.9) and (2.10) are sharp, extremal function being

\[
D^\lambda_z f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \frac{2\xi\beta(1 - \mu)}{(1 + \beta(4\xi - 4\xi\mu - 1))(1 - \alpha)}\right).
\]

\[\square\]
Corollary 2. Let the function $f(z)$ given by (1.2) be in the class $R_{\alpha}[\mu, \beta, \xi]$. Then $D_2^\lambda f(z)$ is included in the disc with center at origin and radius $r_2$ given by,

$$r_2 = \frac{1}{\Gamma(2-\lambda)} \left(1 + \frac{2\xi\beta(1-\mu)}{(1+\beta(4\xi-4\xi\mu-1))(1-\alpha)}\right),$$

where $0 \leq \lambda < 1$.

Finally, we obtain extreme points of $R_{\alpha}[\mu, \beta, \xi]$ by the routine calculation.

Theorem 3. Let

$$f_1(z) = z$$

and

$$f_n(z) = z - \frac{2\beta\xi(1-\mu)}{(n-1) + \beta(2\xi(n-\mu) - (n-1)))C(\alpha, n)} z^n,$$

$n = 2, 3, \cdots$. Then $f \in R_{\alpha}[\mu, \beta, \xi]$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \geq 0$, $\sum_{n=1}^{\infty} \lambda_n = 1$.

References


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