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TYPICALLY REAL HARMONIC FUNCTIONS

Sook Heui Jun

Abstract. In this paper, we study harmonic orientation-preserving univalent mappings defined on $\Delta = \{z : |z| > 1\}$ that are typically real.

1. Introduction

A continuous function $f = u + iv$ defined in a domain $D \subsetneqq \mathbb{C}$ is harmonic in D if u and v are real harmonic in D. Let Σ be the class of all complex-valued, harmonic, orientation-preserving, univalent mappings f defined on $\Delta = \{z : |z| > 1\}$, that are normalized at infinity by $f(\infty) = \infty$. Such functions admit the representation

$$f(z) = h(z) + \overline{g(z)} + Alog|z|,$$

where

$$h(z) = \alpha z + \sum_{k=0}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in Δ and $0 \leq |\beta| < |\alpha|$. In addition, $a = \overline{f_{\overline{z}}}/f_z$ is analytic and satisfies $|a(z)| < 1$. Also one can easily show that $|A|/2 \leq |\alpha| + |\beta|$ by using the bound $|s_1| \leq 1 - |s_0|^2$ for analytic function $a = s_0 + s_1 z^{-1} + \cdots$ in Δ that are bounded by one. By applying an affine post-mapping to f we may normalize f so that $\alpha = 1, \beta = 0$, and $a_0 = 0$. Therefore let Σ' be the set of all harmonic, orientation-preserving, univalent mappings

$$f(z) = h(z) + \overline{g(z)} + Alog|z| \tag{1.1}$$

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of Δ, where

$$h(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in Δ and $A \in \mathbb{C}$. Hengartner and Schober [2] used the representation (1.1) to obtain coefficient bounds and distortion theorems.

In this article, we study functions in Σ' that are typically real.

2. Typically real functions

A function f harmonic in Δ is said to be typically real if $f(z)$ is real if, and only if, z is real. To obtain growth conditions we will further assume that f is orientation-preserving, univalent in Δ. If $f = h + \overline{g} + Alog|z|$ with $f(\infty) = \infty$ we suppose that $|g'(z) + \overline{A}/(2z)| < |h'(z) + A/(2z)|$ and $h(z) \neq g(z)$ for all $z \in \Delta$, $\alpha = 1, \beta = 0,$ and $a_0 = 0$. The class of such functions is denoted by T. The subclass of T with real A is denoted by T_R:

$$T = \{f \in \Sigma' : f \text{ is typically real and } h(z) \neq g(z) \text{ for all } z \in \Delta\}.$$

Theorem 2.1. *Let $f \in \Sigma$. If A and coefficients are real, then f is typically real.*

Proof. For $\overline{f(z)} = f(\overline{z})$ and so $\overline{f(z)} = f(z)$ if, and only if, $z = \overline{z}$ because of the univalence. □

Lemma 2.2. *Let $H(z) = z + \sum_{k=0}^{\infty} s_k z^{-k}$ be analytic and typically real in Δ, with $H(z) \neq 0$ for all $z \in \Delta$. Then the function G defined by*

$$G(\zeta) = \{H(1/\zeta)\}^{-1}$$

is analytic and typically real in the unit disk $\mathbb{D} = \{\zeta : |\zeta| < 1\}$.

Proof. $\overline{G(\zeta)} = G(\zeta)$ if, and only if, $\overline{H(1/\zeta)} = H(1/\zeta)$. $\overline{H(1/\zeta)} = H(1/\zeta)$ if, and only if, $\zeta = \overline{\zeta}$ since H is typically real. Thus G is analytic and typically real in the unit disk $\mathbb{D}$. □

REMARK. Note that $\mathrm{Im}\{G(\zeta)\} > 0$ when $\mathrm{Im}\{\zeta\} > 0$, and $\mathrm{Im}\{G(\zeta)\} < 0$ when $\mathrm{Im}\{\zeta\} < 0$, because $G(0) = 0$ and $G'(0) = 1$ give G these properties near the origin. Also, it is clear that the typically real function $G(\zeta) = \zeta + \sum_{k=2}^{\infty} r_k \zeta^k$ in $\mathbb{D}$ has real coefficients, because $r_k = G^{(k)}(0)/k!$. Thus one can easily show the following Lemma by using the relation $G(\zeta) = \{H(1/\zeta)\}^{-1}$.

LEMMA 2.3. *Let $H(z) = z + \sum_{k=0}^{\infty} s_k z^{-k}$ be analytic and typically real in Δ, with $H(z) \neq 0$ for all $z \in \Delta$. Then $H(z)$ has real coefficients and*

$$\mathrm{Im}\{H(z)\} = \begin{cases} > 0 & \text{if } \mathrm{Im}\{z\} > 0 \\ < 0 & \text{if } \mathrm{Im}\{z\} < 0. \end{cases}$$

LEMMA 2.4. *Let $f \in T_R$. Then*

(1) *the analytic function $h(z) - g(z)$ is typically real and has real coefficients,*

(2) $\mathrm{Im}\{h(re^{i\theta}) - g(re^{i\theta})\} = \begin{cases} > 0 & (\ 0 < \theta < \pi) \\ < 0 & (-\pi < \theta < 0), \end{cases}$

(3) $\mathrm{Im}\{f(re^{i\theta})\} = \begin{cases} > 0 & (\ 0 < \theta < \pi) \\ < 0 & (-\pi < \theta < 0). \end{cases}$

Proof. The analytic function $h(z) - g(z) = z + \sum_{k=1}^{\infty}(a_k - b_k)z^{-k}$ is typically real since $\mathrm{Im}\{f(z)\} = \mathrm{Im}\{h(z) - g(z)\}$ and f is typically real. By Lemma 2.3, $h(z) - g(z)$ has real coefficients and

$$\mathrm{Im}\{f(re^{i\theta})\} = \mathrm{Im}\{h(re^{i\theta}) - g(re^{i\theta})\} = \begin{cases} > 0 & (\ 0 < \theta < \pi) \\ < 0 & (-\pi < \theta < 0). \end{cases}$$

□

Let us mention the following result for the typically real function in the unit disk $\mathbb{D}$ from [1].

LEMMA 2.5[1, THEOREM 2.21]. *If the analytic function*

$$G(\zeta) = \zeta + \sum_{k=2}^{\infty} r_k \zeta^k$$

is typically real in the unit disk $\mathbb{D}$, then $|r_{k+2} - r_k| \leq 2, \quad k = 0, 1, 2, \ldots.$

THEOREM 2.6. *For each f of T_R we have*

$$|a_1 - b_1| \leq 3, \quad |a_2 - b_2| \leq 2, \quad |a_3 - b_3| \leq 8,$$

$$|a_4 - b_4| \leq 12, \quad |a_5 - b_5| \leq 28, \quad |a_6 - b_6| \leq 52.$$

Proof. Let $H(z) = h(z) - g(z) = z + \sum_{k=1}^{\infty} s_k z^{-k}$, where $s_k = a_k - b_k$, and $G(\zeta) = \{H(1/\zeta)\}^{-1}$. Then $G(\zeta)$ is analytic and typically real in $\mathbb{D}$ by Lemma 2.4 and Lemma 2.2. Since

$$\begin{aligned} G(\zeta) = \zeta &- s_1\zeta^3 - s_2\zeta^4 + (s_1^2 - s_3)\zeta^5 + (2s_1s_2 - s_4)\zeta^6 \\ &+ (2s_1s_3 - s_5 + s_2^2 - s_1^3)\zeta^7 + (2s_2s_3 + 2s_1s_4 - s_6 - 3s_1^2s_2)\zeta^8 + \cdots, \end{aligned}$$

we obtain coefficient bounds by Lemma 2.5. □

References

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Department of Mathematics
Seoul Women's University
126 Kongnung 2-dong, Nowon-Gu
Seoul, 139-774, Korea