Kangweon-Kyungki Math. Jour. **8** (2000), No. 2, pp. 147–154

ON FUZZY k-IDEALS IN SEMIRINGS

Seung Il Baik* and Hee Sik Kim

Abstract. In this paper, with the notion of fuzzy k-ideals of semirings, we discuss and review several results described in [4].

1. Introduction

L. A. Zadeh ([10]) introduced the notion of a fuzzy subset μ of a set X as a function from X into the closed unit interval $[0,1]$. The concept of fuzzy subgroups was introduced by A. Rosenfeld ([8]). W. J. Liu ([7]) studied fuzzy ideals in rings. T. K. Dutta and B. K. Biswas ([2, 3]) studied fuzzy ideals, fuzzy prime ideals of semirings, and they defined fuzzy k-ideals and fuzzy prime k-ideals of semirings and characterized fuzzy prime k-ideals of semirings of non-negative integers and determined all its prime k-ideals. Recently, Y. B. Jun, J. Neggers and H. S. Kim ([4]) extended the concept of an L-fuzzy (characteristic) left (resp., right) ideal of a ring to a semiring R, and showed that each level left (resp., right) ideal of an L-fuzzy left (resp., right) ideal μ of R is characteristic iff μ is L-fuzzy characteristic. The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many authors (e.g., [1,5,6,9,11]). The notion of k-ideals and Q-ideals ([1]) were applied to construct quotient semirings. In this paper, with the notion of fuzzy k-ideals of semirings, we discuss and review several results described in [4].

2. Preliminaries

An algebra $(R;+,\cdot)$ is said to be a *semiring* ([11]) if $(R;+)$ and $(R;\cdot)$

Received July 25, 2000.
1991 Mathematics Subject Classification: 16Y60, 04A72.
Key words and phrases: semiring, (fuzzy) k-ideal, characteristic.
$*$ supported by the Catholic University of Korea Research Fund, 2000.

are semigroups satisfying $a\cdot(b+c)=a\cdot b+a\cdot c$ and $(b+c)\cdot a=b\cdot a+c\cdot a$ for all $a,b,c\in R$. A semiring R may have an identity 1, defined by $1\cdot a=a=a\cdot 1$, and a zero 0, defined by $0+a=a=a+0$ and $a\cdot 0=0=0\cdot a$ for all $a\in R$.

From now on we write R and S for semirings. A non-empty subset I of R is said to be a *left* (resp., *right*) *ideal* if $x,y\in I$ and $r\in R$ imply that $x+y\in I$ and $rx\in I$ (resp., $xr\in I$). If I is both left and right ideal of R, we say I is a two-sided ideal, or simply, ideal of R. A left ideal I of a semiring R is said to be a *left k-ideal* if $a\in I$ and $x\in R$ and if $a+x\in I$ or $x+a\in I$ then $x\in I$. Right k-ideal is defined dually, and two-sided k-ideal or simply a k-ideal is both a left and a right k-ideal.

A mapping $f:R\to S$ is said to be a *homomorphism* if $f(x+y)=f(x)+f(y)$ and $f(xy)=f(x)f(y)$ for all $x,y\in R$. We note that if $f:R\to S$ is an onto homomorphism and I is a left (resp., right) ideal of R, then $f(I)$ is a left (resp., right) ideal of S.

DEFINITION 2.1 ([2]). A fuzzy subset μ of a semiring R is said to be a *fuzzy left* (resp.,*right*) *ideal* of R if $\mu(x+y)\geq\min\{\mu(x),\mu(y)\}$ and $\mu(xy)\geq\mu(y)$ (resp., $\mu(xy)\geq\mu(x)$) for all $x,y\in R$. μ is a fuzzy ideal of R if it is both a fuzzy left and a fuzzy right ideal of R.

DEFINITION 2.2 ([3]). A fuzzy ideal μ of a semiring R is said to be a *fuzzy k-ideal* of R if

$$\mu(x)\geq\min\{\max\{\mu(x+y),\mu(y+x)\},\mu(y)\}$$

for all $x,y\in R$. If R is additively commutative then the condition reduces to $\mu(x)\geq\min\{\mu(x+y),\mu(y)\}$ for all $x,y\in R$.

Note that every fuzzy ideal of a ring is a fuzzy k-ideal.

EXAMPLE 2.3 ([3]). Let μ be a fuzzy subset of the semiring N of natural numbers defined by

$$\mu(x):=\begin{cases}0.3, & \text{if } x \text{ is odd,}\\ 0.5, & \text{if } x \text{ is non-zero even,}\\ 1, & \text{if } x=0.\end{cases}$$

Then μ is a fuzzy k-ideal of N.

EXAMPLE 2.4 ([3]). Let μ be a fuzzy subset of the semiring N of natural numbers defined by

$$\mu(x) := \begin{cases} 1, & \text{if } 7 \le x, \\ 0.5, & \text{if } 5 \le x < 7, \\ 0, & \text{if } 0 \le x < 5. \end{cases}$$

Then it is easy to show that μ is a fuzzy ideal of N, but not a fuzzy k-ideal of N.

PROPOSITION 2.5 ([3]). *Let I be a non-empty subset of a semiring R and λ_I the characteristic function of I. Then I is a k-ideal of R if and only if λ_I is a fuzzy k-ideal of R.*

3. Main Results

Y. B. Jun, J. Neggers and H. S. Kim ([4]) studied L-fuzzy ideals in semirings, and T. K. Dutta and B. K. Biswas ([3]) defined the notion of fuzzy k-ideals in semirings. With the notion of fuzzy k-ideals of semirings we discuss and review several results described in [4].

PROPOSITION 3.1. *A fuzzy subset μ of R is a fuzzy left (resp., right) k-ideal of R if and only if, for any $t \in [0,1]$ such that $\mu_t \neq \emptyset$, μ_t is a left (resp., right) k-ideal of R, where $\mu_t = \{x \in R | \mu(x) \ge t\}$, which is called a level subset of μ.*

Proof. It was proved that a fuzzy subset μ is a fuzzy left (resp., right) ideal of R if and only if for any $t \in [0,1]$ such that $\mu_t \neq \emptyset$, μ_t is a left (resp., right) ideal of R (see [4]). Assume that μ is a fuzzy k-ideal of R. Suppose that $a \in \mu_t$ and $x \in R$, and $a + x \in \mu_t$ or $x + a \in \mu_t$. Then $\mu(a) \ge t, \mu(a+x) \ge t$ or $\mu(x+a) \ge t$, and hence $\max\{\mu(a+x), \mu(x+a)\} \ge t$. Since μ is a fuzzy k-ideal of R, $\mu(x) \ge \min\{\max\{\mu(a+x), \mu(x+a)\}, \mu(a)\}$, i.e., $x \in \mu_t$. Hence μ_t is a k-ideal of R.

Conversely, assume μ_t is a k-ideal of R, for any $t \in [0,1]$ with $\mu_t \neq \emptyset$. For any $x, a \in R$, let $\mu(a) = t_1, \mu(x+a) = t_2, \mu(a+x) = t_3$ ($t_i \in [0,1]$). If we let $t := \min\{\max\{t_2, t_3\}, t_1\}$, then $a \in \mu_t$ and $a + x \in \mu_t$ or $x + a \in \mu_t$. Since μ_t is a k-ideal of R, we have $x \in \mu_t$, i.e.,

$\mu(x) \geq \min\{\max\{\mu(x+a), \mu(a+x)\}, \mu(a)\}$, proving that μ is a fuzzy k-ideal of R. □

Note that if μ is a fuzzy left (resp., right) k-ideal of R then the set $R_\mu := \{x \in R | \mu(x) \geq \mu(0)\}$ is a left (resp., right) k-ideal of R.

THEOREM 3.2. *Let I be any left (resp., right) k-ideal of R. Then there exists a fuzzy left (resp., right) k-ideal μ of R such that $\mu_t = I$ for some $t \in [0,1]$.*

Proof. If we define a fuzzy subset of R by

$$\mu(x) := \begin{cases} t & \text{if } x \in I, \\ 0 & \text{otherwise} \end{cases}$$

for some $t \in [0,1]$, then it follows that $\mu_t = I$. For a given $s \in [0,1]$ we have

$$\mu_s = \begin{cases} \mu_0 \ (= R) & \text{if } s = 0, \\ \mu_t \ (= I) & \text{if } s \leq t, \\ \emptyset & \text{if } t < s \leq 1. \end{cases}$$

Since I and R itself are left (resp., right) k-ideals of R, it follows that every non-empty level subset μ_s of μ is a left (resp., right) k-ideal of R. By Proposition 3.1, μ is a fuzzy left (resp., right) k-ideal of R, proving the theorem. □

Let μ and ν be fuzzy subsets of R. We denote that $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in X$, and $\mu \subset \nu$ if and only if $\mu \subseteq \nu$ and $\mu \neq \nu$.

THEOREM 3.3. *Let μ be a fuzzy left (resp., right) k-ideal of R. Then two level left (resp., right) k-ideals μ_s, μ_t (with $s < t$ in $[0,1]$) of μ are equal if and only if there is no $x \in R$ such that $s \leq \mu(x) < t$.*

Proof. Suppose $s < t$ in $[0,1]$ and $\mu_s = \mu_t$. If there exists an $x \in R$ such that $s \leq \mu(x) < t$, then μ_t is a proper subset of μ_s, a contradiction. Conversely, suppose that there is no $x \in R$ such that $s \leq \mu(x) < t$. Note that $s < t$ implies $\mu_t \subseteq \mu_s$. If $x \in \mu_s$, then $\mu(x) \geq s$, and so $\mu(x) \geq t$ because $\mu(x) \not< t$. Hence $x \in \mu_t$, and $\mu_s = \mu_t$. This completes the proof. □

Given a fuzzy k-ideal μ of R we denote by $\mathrm{Im}(\mu)$ the image set of μ.

THEOREM 3.4. *Let μ be a fuzzy left (resp., right) k-ideal of R. If* $\mathrm{Im}(\mu)= \{t_1, t_2, ..., t_n\}$, *where $t_1 < t_2 < ... < t_n$, then the family of left (resp., right) k-ideals μ_{t_i} $(i = 1, ..., n)$ constitutes the collection of all level left (resp., right) ideals of μ.*

Proof. If $t \in [0, 1]$ with $t < t_1$, then $\mu_{t_1} \subseteq \mu_t$. Since $\mu_{t_1} = R$, we have $\mu_t = R$ and $\mu_t = \mu_{t_1}$. If $t \in [0, 1]$ with $t_i < t < t_{i+1}$ $(1 \leq i \leq n - 1)$, then there is no $x \in R$ such that $t \leq \mu(x) < t_{i+1}$. It follows from Theorem 3.3 that $\mu_t = \mu_{t_{i+1}}$. This shows that for any $t \in L$ with $t \leq \mu(0)$, the level left (resp., right) ideal μ_t is in $\{\mu_{t_i} | 1 \leq i \leq n\}$. This completes the proof. □

Given any two sets R and S, let μ be a fuzzy subset of R and let $f : R \to S$ be any function. We define a fuzzy subset ν on S by

$$\nu(y) := \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, y \in S, \\ 0 & \text{otherwise}, \end{cases}$$

and we call ν the *image* of μ under f, written $f(\mu)$. For any fuzzy subset ν on $f(R)$, we define a fuzzy subset μ on R by $\mu(x) := \nu(f(x))$ for all $x \in R$, and we call μ the *preimage* of ν under f which is denoted by $f^{-1}(\nu)$.

THEOREM 3.5. *An onto homomorphic preimage of a fuzzy left (resp., right) k-ideal is a fuzzy left (resp., right) k-ideal.*

Proof. Let $f : R \to S$ be an onto homomorphism. Let ν be a fuzzy left (resp., right) k-ideal on S and let μ be the preimage of ν under f. Then it was proved that μ is a fuzzy left (resp., right) ideal of R ([4]). For any $x, y \in S$, we have

$$\begin{aligned} \mu(x) &= \nu(f(x)) \\ &\geq \min\{\max\{\nu(f(x) + f(y)), \nu(f(y) + f(x))\}, \nu(f(y))\} \\ &= \min\{\max\{\nu(f(x + y)), \nu(f(y + x))\}, \nu(f(y))\} \\ &= \min\{\max\{\mu(x + y), \mu(y + x)\}, \mu(y)\}, \end{aligned}$$

proving that μ is a fuzzy left (resp., right) k-ideal of R. □

Proposition 3.6 ([4]). *Let f be a mapping from a set X to a set Y, and let μ be a fuzzy subset of X. Then for every $t \in (0,1]$,*

$$(f(\mu))_t = \bigcap_{0<s<t} f(\mu_{t-s}).$$

Theorem 3.7. *Let $f : R \to S$ be an onto homomorphism and let μ be a fuzzy left (resp., right) k-ideal of R. Then the homomorphic image $f(\mu)$ of μ under f is a fuzzy left (resp., right) k-ideal of S.*

Proof. In view of Proposition 3.1 it is sufficient to show that each non-empty level subset of $f(\mu)$ is a left (resp., right) k-ideal of S. Let $(f(\mu))_t$ be a non-empty level subset of $f(\mu)$ for every $t \in [0,1]$. If $t = 0$ then $(f(\mu))_t = S$. Assume $t \neq 0$. By Proposition 3.6, $(f(\mu))_t = \bigcap_{0<s<t} f(\mu_{t-s})$. Hence $f(\mu_{t-s})$ is non-empty for each $0 < s < t$, and so μ_{t-s} is a non-empty level subset of μ for every $0 < s < t$. Since μ is a fuzzy left (resp., right) k-ideal of R, it follows from Proposition 3.1 that μ_{t-s} is a left (resp., right) k-ideal of R. Since f is an onto homomorphism, $f(\mu_{t-s})$ is a left (resp., right) k-ideal of S. Hence $(f(\mu))_t$ being an intersection of a family of left (resp., right) k-ideals is also a left (resp., right) k-ideal of S. The proof is complete. □

Definition 3.8. A left (resp., right) k-ideal I of R is said to be *characteristic* if $f(I) = I$ for all $f \in \mathrm{Aut}(R)$, where $\mathrm{Aut}(R)$ is the set of all automorphisms of R. A fuzzy left (resp., right) k-ideal μ of R is said to be a *fuzzy characteristic* if $\mu(f(x)) = \mu(x)$ for all $x \in R$ and $f \in \mathrm{Aut}(R)$.

Theorem 3.9. *Let μ be a fuzzy left (resp., right) k-ideal of R and let $f : R \to R$ be an onto homomorphism. Then the mapping $\mu^f : R \to [0,1]$, defined by $\mu^f(x) := \mu(f(x))$ for all $x \in R$, is a fuzzy left (resp., right) k-ideal of R.*

Proof. It was proved that μ^f is a fuzzy left (resp., right) k-ideal of R ([4]). For any $x, y \in R$, we have

$$\begin{aligned}
\mu^f(x) &= \mu(f(x)) \\
&\geq \min\{\max\{\mu(f(x)+f(y)),\, \mu(f(y)+f(x))\},\, \mu(f(y))\} \\
&= \min\{\max\{\mu(f(x+y)),\, \mu(f(y+x))\},\, \mu(f(y))\} \\
&= \min\{\max\{\mu^f(x+y),\, \mu^f(y+x)\},\, \mu^f(x)\},
\end{aligned}$$

proving that μ^f is a fuzzy left (resp., right) k-ideal of R. □

THEOREM 3.10. *If μ is a fuzzy characteristic left (resp., right) k-ideal of R, then each level left (resp., right) k-ideal of μ is characteristic.*

Proof. Let μ be a fuzzy characteristic left (resp., right) k-ideal of R and let $f \in \mathrm{Aut}(R)$. For any $t \in [0,1]$, if $y \in f(\mu_t)$, then $\mu(y) = \mu(f(x)) = \mu(x) \geq t$ for some $x \in \mu_t$ with $y = f(x)$. It follows that $y \in \mu_t$. Conversely, if $y \in \mu_t$, then $t \leq \mu(y) = \mu(f(x)) = \mu(x)$ for some $x \in R$ with $y = f(x)$. It follows that $y \in f(\mu_t)$. This completes the proof. □

To prove the converse of Theorem 3.10, we need the following lemma.

LEMMA 3.11. *Let μ be a fuzzy left (resp., right) k-ideal of R and let $x \in R$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$.*

Proof. Straightforward. □

THEOREM 3.12. *Let μ be a fuzzy left (resp., right) k-ideal of R. If each level left (resp. right) k-ideal of μ is characteristic, then μ is fuzzy characteristic.*

Proof. Let $x \in R$ and $f \in \mathrm{Aut}(R)$. If $\mu(x) = t \in [0,1]$, then by Lemma 3.11 $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$. Since each level left (resp., right) k-ideal of μ is characteristic, $f(x) \in f(\mu_t) = \mu_t$. Assume $\mu(f(x)) = s > t$. Then $f(x) \in \mu_s = f(\mu_s)$. Since f is one-to-one, it follows that $x \in \mu_s$, a contradiction. Hence $\mu(f(x)) = t = \mu(x)$, showing that μ is fuzzy characteristic. □

References

1. P. J. Allen, *A fundamental theorem of homomorphisms for semirings*, Proc. Amer. Math. Soc. **21** (1969), 412 - 416.
2. T. K. Dutta and B. K. Biswas, *Fuzzy prime ideals of a semiring*, Bull. Malaysian Math. Soc. **17** (1994), 9-16.
3. T. K. Dutta and B. K. Biswas, *Fuzzy k-ideals of semirings*, Bull. Calcutta Math. Soc. **87** (1995), 91-96.
4. Y. B. Jun, J. Neggers and H. S. Kim, *On L-fuzzy ideals in semirings* I, Czech. Math. J. **48** (1998), 669-675.

5. H. S. Kim, *On quotient semiring and extension of quotient halfring*, Comm. Korean Math. Soc. **4** (1989), 17-22.
6. D. R. Latorre, *On h-ideals and k-ideals in hemirings*, Pub. Math. Debrecen **12** (1965), 219-226.
7. W. J. Liu, *Fuzzy invariants subgroups and fuzzy ideals*, Fuzzy Sets and Sys. **8** (1987), 133-139.
8. A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512-517.
9. M. K. Sen and M. R. Adhikari, *On maximal k-ideals of semirings*, Proc. Amer. Math. Soc. **118** (1993), 699-703.
10. L. A. Zadeh, *Fuzzy sets*, Inform. and Control **8** (1965), 338 - 353.
11. J. Zeleznekow, *Regular semirings*, Semigroup Forum **23** (1981), 119-136.

Seung Il Baik
Dept. of Mathematics
The Catholic University of Korea
Puchon, 422-743, Korea

Hee Sik Kim
Dept. of Mathematics
Hanyang University
Seoul, 133-791, Korea
E-mail: heekim@email.hanyang.ac.kr