LINEAR FUNCTIONALS ON $\mathcal{O}_n$ AND PRODUCT PURE STATES OF UHF

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ABSTRACT. For a sequence $\{\eta_m\}_m$ of unit vectors in $\mathbb{C}^n$, we consider the associated linear functional ω on the Cuntz algebra $\mathcal{O}_n$. We show that the restriction $\omega|_{\mathrm{UHF}_n}$ is the product pure state of a subalgebra UHF_n of $\mathcal{O}_n$ such that $\omega|_{\mathrm{UHF}_n} = \otimes\omega_m$ with $\omega_m(\cdot) = < \cdot\eta_m, \eta_m >$. We study product pure states of UHF and obtain a concrete description of them in terms of unit vectors. We also study states of UHF_n which is the restriction of the linear functionals on $\mathcal{O}_n$ associated to a fixed unit vector in $\mathbb{C}^n$.

1. Introduction

In [4], J. Cuntz defined the *Cuntz algebra* $\mathcal{O}_n$ by the C^*-algebra generated by $n = 2, 3, \cdots$ isometries $s_1, s_2, \cdots, s_n$ satisfying the Cuntz relations:

$$s_i^* s_j = \delta_{ij} 1 \ \text{ and } \ \sum_{i=1}^{n} s_i s_i^* = 1.$$

A UHF *algebra* is a uniformly hyperfinite algebra and a UHF_n algebra is a UHF algebra with Glimm type n^∞. We consider UHF_n as a subalgebra of $\mathcal{O}_n$.

One of the recent topics of the study of $\mathcal{O}_n$ is that of representations of $\mathcal{O}_n$ and UHF_n(see [1], [2], [3], [7]). In detail, there is a correspondence between representations of $\mathcal{O}_n$ and endomorphisms of $\mathcal{B}(\mathcal{H})$ of Powers index n, where $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on a Hilbert space $\mathcal{H}$. However, to study of representations of the Cuntz algebra $\mathcal{O}_n$ is not so easy. Because the simple C^*-algebra $\mathcal{O}_n$ is the famous example whose representations are bad. In general, the

representation of $\mathcal{O}_n$, except with severe conditions in which cases give irreducible representations, is decomposed into irreducible subrepresentations of it. As is known, representations are related to states and irreducible representations of a C^*-algebra correspond to pure states of it by GNS constructions. Thus, our purpose is to study of pure states of $\mathcal{O}_n$ and UHF_n which give irreducible representations of these C^*-algebras . More generally, we concern pure states of a uniformly hyperfinite algebra UHF.

Glimm [5] showed that a UHF algebra is a norm separable C^*-algebra which is the norm closure of an increasing sequence of type I_{n_m}-factors. In other words, UHF can be identified with $\overset{\infty}{\underset{m=1}{\otimes}} M_{n_m}$, where M_{n_m} is an $n_m \times n_m$ matrix algebra, and UHF_n is the UHF algebra $\overset{\infty}{\underset{m=1}{\otimes}} M_n$. In addition, a state ρ of UHF is a product state if ρ satisfies that $\rho(xy) = \rho(x)\rho(y)$ for $x \in I \otimes \cdots \otimes I \otimes M_{n_i} \otimes I \otimes \cdots$ and $y \in I \otimes \cdots \otimes I \otimes M_{n_j} \otimes I \otimes \cdots$ $(i \neq j)$ and Powers [8] studied product states of UHF.

On the other hand, Bratelli, Jorgensen, and Price [3] introduced a state ω_η of $\mathcal{O}_n$ for a fixed unit vector $\eta = (\eta^1, \eta^2, \cdots, \eta^n)$ in $\mathbb{C}^n$, defined by

$$\omega_\eta(s_{i_1} \cdots s_{i_k} s_{j_l}^* \cdots s_{j_1}^*) = \eta^{i_1}\eta^{i_2}\cdots\eta^{i_k}\overline{\eta^{j_l}}\overline{\eta^{j_{l-1}}}\cdots\overline{\eta^{j_1}}$$

and called it the *Cuntz state* which is a pure state of $\mathcal{O}_n$. We know that the restriction $\omega_\eta|_{\mathrm{UHF}_n}$ of the Cuntz state ω_η to a subalgebra UHF_n of $\mathcal{O}_n$ becomes a product pure state of UHF_n.

In this paper, we start with the natural generalization of the Cuntz state to a linear functional associated to a sequence of unit vectors. We obtain the description of product pure states of UHF_n related to sequences of unit vectors in $\mathbb{C}^n$. More generally, we study all product pure states of a UHF algebra in terms of unit vectors. We know that the extensions of these product pure states of UHF_n to $\mathcal{O}_n$ become linear functionals on $\mathcal{O}_n$. It is known that product pure states of a UHF_n algebra have extensions to $\mathcal{O}_n$ as states (see [3]). Furthermore, we study states of UHF_n which come from states of $\mathcal{O}_n$.

Throughout this paper, the inner product $< \xi, \eta >$ of two vectors $\xi = (\xi^1, \cdots, \xi^n)$, $\eta = (\eta^1, \cdots, \eta^n) \in \mathbb{C}^n$ is given by $< \xi, \eta > = \sum \xi^i \overline{\eta^i}$.

2. The relation between Product Pure States of UHF and unit vectors

In this section, we describe all product pure states of a UHF algebra in terms of unit vectors.

We note that a pure state of a matrix algebra is a vector state. So, for a pure state ρ of a matrix algebra M_n, there exists a unit vector $\eta \in \mathbb{C}^n$ such that $\rho(A) =< A\eta, \eta >$ for all $A \in M_n$.

We have already noted that the restriction $\omega_\eta|_{\mathrm{UHF}_n}$ of the Cuntz state ω_η to a subalgebra UHF_n is a product pure state of UHF_n. But, in general, the restriction of a linear functional ω on $\mathcal{O}_n$ to UHF_n, even when ω is a pure state of $\mathcal{O}_n$, may not be a pure state. We investigate a certain linear functional on $\mathcal{O}_n$ whose restriction to UHF_n becomes a pure state.

For our discussion, we define the linear functional on the Cuntz algebra $\mathcal{O}_n$ associated to a sequence of unit vectors in $\mathbb{C}^n$. Recall that the Cuntz algebra $\mathcal{O}_n$ is the closure of the linear span of operators $s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_l}^* s_{j_{l-1}}^* \cdots s_{j_1}^*$ for $k, l = 0, 1, \cdots$. As a subalgebra of the Cuntz algebra $\mathcal{O}_n$, UHF_n is the closure of the linear span of operators $s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_k}^* s_{j_{k-1}}^* \cdots s_{j_1}^*$ for $k = 0, 1, \cdots$.

DEFINITION 2.1. For a sequence $\{\eta_m = (\eta_m^1, \cdots, \eta_m^n)\}_m$ of unit vectors in $\mathbb{C}^n$, the *associated linear functional* ω on $\mathcal{O}_n$ is defined by

$$\omega(s_{i_1} s_{i_2} \cdots s_{i_k} s_{j_l}^* s_{j_{l-1}}^* \cdots s_{j_1}^*) = \eta_1^{i_1} \eta_2^{i_2} \cdots \eta_k^{i_k} \overline{\eta_l^{j_l}} \, \overline{\eta_{l-1}^{j_{l-1}}} \cdots \overline{\eta_1^{j_1}}.$$

In fact, for a constant sequence $\{\eta = (\eta^1, \cdots, \eta^n)\}_m$ of a unit vector in $\mathbb{C}^n$, the associated linear functional ω_η on $\mathcal{O}_n$ is just the Cuntz state.

Here we prove that the restriction of the associated linear functional on $\mathcal{O}_n$ to UHF_n is a pure state. But, it is not known yet that whether these linear functionals are states of $\mathcal{O}_n$ or not.

In the following, a UHF_n algebra can be identified with a UHF algebra $\overset{\infty}{\underset{m=1}{\otimes}} M_n$ which is the uniform closure of finite linear combinations of the form $A_1 \otimes A_2 \otimes \cdots$, where each A_m acts on a fixed n-dimensional Hilbert space and all but finitely many of the A_m's are identity.

THEOREM 2.2. *Let $\{\eta_m\}_m$ be a sequence of unit vectors $\eta_m \in \mathbb{C}^n$ and ω the associated linear functional on $\mathcal{O}_n$. Then the restriction*

$\omega|_{\mathrm{UHF}_n}$ of ω is the product pure state of UHF_n such that $\omega|_{\mathrm{UHF}_n} = \overset{\infty}{\underset{m=1}{\otimes}} \omega_m$, where ω_m is the vector state of M_n defined by $\omega_m(\cdot) = < \cdot \eta_m, \eta_m >$.

Proof. For a sequence $\{\eta_m\}_m$ of unit vectors $\eta_m \in \mathbb{C}^n$, consider vector states ω_m of M_n defined by $\omega_m(A) = < A\eta_m, \eta_m >$ for $A \in M_n$. Then we get a state $\overset{\infty}{\underset{m=1}{\otimes}} \omega_m$ of $\overset{\infty}{\underset{m=1}{\otimes}} M_n$.

With the identification $\mathrm{UHF}_n = \overset{\infty}{\underset{m=1}{\otimes}} M_n$, since $s_{i_1} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_1}^*$ in UHF_n can be identified with $E_{i_1 j_1} \otimes \cdots \otimes E_{i_k j_k} \otimes I \otimes \cdots$, where E_{ij} is the matrix in M_n whose (i,j)-component is 1 and the others are 0, we have

$$(\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_k \otimes \cdots)(E_{i_1 j_1} \otimes E_{i_2 j_2} \otimes \cdots \otimes E_{i_k j_k} \otimes I \otimes \cdots)$$
$$= \eta_1^{i_1} \overline{\eta_1^{j_1}} \eta_2^{i_2} \overline{\eta_2^{j_2}} \cdots \eta_k^{i_k} \overline{\eta_k^{j_k}}.$$

Note that $\omega|_{\mathrm{UHF}_n}(s_{i_1} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_2}^* s_{j_1}^*) = \eta_1^{i_1} \eta_2^{i_2} \cdots \eta_k^{i_k} \overline{\eta_k^{j_k}} \cdots \overline{\eta_2^{j_2}} \overline{\eta_1^{j_1}}$.

Therefore, we conclude that $\omega|_{\mathrm{UHF}_n} = \overset{\infty}{\underset{m=1}{\otimes}} \omega_m$. Since the infinite tensor product of vector states is pure, it is straightforward to check that $\overset{\infty}{\underset{m=1}{\otimes}} \omega_m$ is a product pure state of $\overset{\infty}{\underset{m=1}{\otimes}} M_n$, which completes the proof. $\qquad\square$

Thanks to Theorem 2.2 above, for the linear functional ω_η on $\mathcal{O}_n$ associated to a constant sequence $\{\eta\}_m$ of a fixed unit vector $\eta \in \mathbb{C}^n$, the restriction $\omega_\eta|_{\mathrm{UHF}_n}$ is the infinite tensor product of a vector state $< \cdot \eta, \eta >$ which is a product pure state of M_n and so $\omega_\eta|_{\mathrm{UHF}_n}$ is a pure state of UHF_n.

The following is an easy result but it is useful later. For the sake of completeness, we give a proof.

LEMMA 2.3. *Let ρ be a product pure state of UHF. Then there exist pure states ρ_m of M_{n_m} for $m = 1, 2, \cdots$ such that $\rho = \overset{\infty}{\underset{m=1}{\otimes}} \rho_m$.*

Proof. By using the identification $\mathrm{UHF} = \overset{\infty}{\underset{m=1}{\otimes}} M_{n_m}$, for a product pure state ρ of a UHF algebra, if we let ρ_m be a state of M_{n_m} for

$m = 1, 2, \cdots$ defined by

$$\rho_m(A) = \rho|_{I \otimes \cdots I \otimes M_{n_m} \otimes I \otimes \cdots}(1 \otimes \cdots \otimes 1 \otimes A \otimes 1 \otimes \cdots), \quad A \in M_{n_m},$$

then it is straightforward to see that $\rho = \overset{\infty}{\underset{m=1}{\otimes}} \rho_m$.

To prove that ρ_i's are pure states, suppose that there exists j such that ρ_j is not pure. Since a state is the linear combination of pure states, there exist two distinct states θ_j and θ'_j such that $\rho_j = \alpha\theta_j + (1-\alpha)\theta'_j$ for some α, $0 < \alpha < 1$. Let us consider two distinct states given by

$$\theta = \rho_1 \otimes \cdots \otimes \rho_{j-1} \otimes \theta_j \otimes \rho_{j+1} \otimes \cdots$$

and

$$\theta' = \rho_1 \otimes \cdots \otimes \rho_{j-1} \otimes \theta'_j \otimes \rho_{j+1} \otimes \cdots.$$

Then we have $\rho = \alpha\theta + (1-\alpha)\theta'$ which is not a pure state. The proof is completed.   □

The preceding lemma asserts that a product pure state ρ of UHF is an infinite tensor product of vector states of matrix algebras.

Using Lemma 2.3, we obtain the following result which describes all product pure states of UHF in terms of unit vectors. In the following, the equivalent class $[\eta]$ of a unit vector $\eta \in \mathbb{C}^n$ denotes the set $\{\lambda\eta | \lambda \in \mathbb{C}, |\lambda| = 1\}$.

THEOREM 2.4. *There exists a one-to-one correspondence between the set of all product pure states of a UHF algebra and that of all sequences $\{[\eta_m]\}_m$ of equivalent classes $[\eta_m]$ of unit vectors $\eta_m \in \mathbb{C}^{n_m}$.*

Proof. Let ρ be a product pure state of UHF. By Lemma 2.3, ρ is the form of $\overset{\infty}{\underset{m=1}{\otimes}} \rho_m$, where ρ_m's are pure states of M_{n_m} and so there exist unit vectors $\eta_m \in \mathbb{C}^{n_m}$ with $\rho_m(\cdot) = < \cdot\eta_m, \eta_m >$.

Since $< \cdot\eta_m, \eta_m > = < \cdot\lambda\eta_m, \lambda\eta_m >$ for any $\lambda \in \mathbb{C}, |\lambda| = 1$, we get a sequence $\{[\eta_m]\}_m$ with unit vectors $\eta_m \in \mathbb{C}^{n_m}$ corresponding to ρ.

Conversely, for any sequence $\{[\eta_m]\}_m$ with unit vectors $\eta_m \in \mathbb{C}^{n_m}$, let $\rho(\cdot) = \otimes < \cdot\eta_m, \eta_m >$ be an infinite tensor product of vector states. Since the tensor product of vector states is a product pure state of UHF (see [6]), ρ is pure.   □

Since $\text{UHF}_n = \overset{\infty}{\underset{m=1}{\otimes}} M_n$, for a product pure state ρ of UHF_n, there exist pure states ρ_m of M_n with $\rho = \overset{\infty}{\underset{m=1}{\otimes}} \rho_m$. ¿From Theorem 2.4, we obtain a one-to-one correspondence between the set of all product pure states of UHF_n and that of all sequences $\{[\eta_m]\}_m$ of equivalent classes $[\eta_m]$ of unit vectors $\eta_m \in \mathbb{C}^n$.

3. States of UHF_n related to linear functionals on $\mathcal{O}_n$

We examine linear functionals on a simple infinite C^*-algebra $\mathcal{O}_n$ and states of UHF_n related to linear functionals on $\mathcal{O}_n$. For $2 \leq n < \infty$, the Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by isometries $s_1, s_2, \cdots, s_n$ satisfying the Cuntz relations. It turns out that for a state ρ of $\mathcal{O}_n$, the vector $(\rho(s_1), \cdots, \rho(s_n))$ may not be a unit vector, even when ρ is pure.

Note further that for the linear functional ω_η on $\mathcal{O}_n$ associated to a constant sequence $\{[\eta]\}_m$ of a fixed unit vector $\eta \in \mathbb{C}^n$, ω_η satisfies $\omega_\eta(s_i s_j) = \omega_\eta(s_i)\omega_\eta(s_j)$ and $\omega_\eta(s_i^*) = \overline{\omega_\eta(s_i)}$, but it can not be a homomorphism.

In general, for a linear functional ρ on $\mathcal{O}_n$, even when it is a state, $\rho(s_i s_j) = \rho(s_i)\rho(s_j)$ may not be true. In fact, the following says that a state ρ of $\mathcal{O}_n$ can not have homomorphic property of $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in \mathcal{O}_n$.

PROPOSITION 3.1. *Let ρ be a linear functional on $\mathcal{O}_n$. If ρ is a state of $\mathcal{O}_n$, then ρ is not a homomorphism.*

Proof. For a state ρ of $\mathcal{O}_n$, suppose that ρ is a homomorphism. Then $ker\rho$ is an ideal of $\mathcal{O}_n$ containing $0 \in \mathcal{O}_n$. Since the Cuntz algebra $\mathcal{O}_n$ is simple, the ideal $ker\rho$ must be $\{0\}$ or $\mathcal{O}_n$. But the facts that $\rho(\rho(s_2)s_1 - \rho(s_1)s_2) = 0$ and $\rho(1) = 1$ give $ker\rho \neq \{0\}$ and $ker\rho \neq \mathcal{O}_n$, respectively, which is a contradiction. □

Recall that a state of a C^*-algebra is a positive linear functional of norm one and the linear functional on $\mathcal{O}_n$ associated to a constant sequence of a fixed unit vector in $\mathbb{C}^n$ is a state of $\mathcal{O}_n$. Here we report on the Cuntz state ω_η which is a pure state of $\mathcal{O}_n$ and the restriction $\omega_\eta|_{\text{UHF}_n}$ is the product pure state of UHF_n. In fact, the Cuntz state ω_η

is the linear functional on $\mathcal{O}_n$ associated to a constant sequence $\{\eta\}_m$ of a fixed unit vector $\eta \in \mathbb{C}^n$. If we consider the restriction $\omega_\eta|_{\mathrm{UHF}_n}$, then ω_η is the natural extension of $\omega_\eta|_{\mathrm{UHF}_n}$ and it is a state of $\mathcal{O}_n$ associated to a constant sequence. Thus our concern is states of UHF_n which come from linear functionals on $\mathcal{O}_n$ associated to a fixed unit vector in $\mathbb{C}^n$.

In the following, we define a UHF_n-state such as the restriction $\omega_\eta|_{\mathrm{UHF}_n}$ of the Cuntz state ω_η to UHF_n.

DEFINITION 3.2. A state ρ of a UHF_n algebra is called a UHF_n-*state* if there exists a constant sequence $\{\eta\}_m$ of a unit vector $\eta \in \mathbb{C}^n$ such that ρ is the restriction of the associated linear functional ω_η on $\mathcal{O}_n$ to UHF_n.

We note here that for the linear functional ω_η on $\mathcal{O}_n$ associated to a constant sequence $\{\eta\}_m$ of a fixed unit vector $\eta \in \mathbb{C}^n$, the restriction $\omega_\eta|_{\mathrm{UHF}_n}$ is the product pure state with $\omega_\eta|_{\mathrm{UHF}_n} = \otimes\omega_m$, where $\omega_m(\cdot) = <\cdot\eta, \eta>$ for $m = 1, 2, \cdots$. Thus we conclude that a UHF_n-state defined in Definition 3.2 corresponds to a constant sequence of a fixed unit vector in $\mathbb{C}^n$.

From the definition of a UHF_n-state, we know that a UHF_n-state has an extension to $\mathcal{O}_n$ associated to a constant sequence of a unit vector. But the following example shows that a UHF_n-state may have another extension to $\mathcal{O}_n$ whose corresponding sequence of unit vectors is not constant.

EXAMPLE 3.3. For a unit vector $\eta = (\eta^1, \eta^2) \in \mathbb{C}^2$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$, let ω_η and $\omega_{\lambda\eta}$ be two linear functionals on $\mathcal{O}_2$ associated to constant sequences $\{\eta\}_m$ and $\{\lambda\eta\}_m$, respectively. If we let ω be a linear functional on $\mathcal{O}_2$ given by $\omega = \frac{1}{2}(\omega_\eta + \omega_{\lambda\eta})$. Then we obtain UHF_2-states $\omega|_{\mathrm{UHF}_2}$, $\omega_\eta|_{\mathrm{UHF}_2}$, and $\omega_{\lambda\eta}|_{\mathrm{UHF}_2}$ with their extension ω, ω_η, and $\omega_{\lambda\eta}$, respectively. Elementary computations give $\omega|_{\mathrm{UHF}_2} = \omega_\eta|_{\mathrm{UHF}_2} = \omega_{\lambda\eta}|_{\mathrm{UHF}_2}$. However, for an extension ω of a UHF_2-state $\omega|_{\mathrm{UHF}_2}$, the corresponding sequence of unit vectors cannot be constant.

To see this, let us consider $\omega(s_1) = \frac{1+\lambda}{2}\eta^1$ and $\omega(s_1^2) = \frac{1+\lambda^2}{2}(\eta^1)^2$. Suppose that the sequence corresponding to ω is constant sequence of a fixed unit vector. Then we have $\omega(s_1)^2 = \omega(s_1^2)$ and so $(\frac{1+\lambda}{2})^2 = \frac{1+\lambda^2}{2}$ which implies $\lambda = 1$.

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