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COHOMOLOGY GROUPS OF CIRCULAR UNITS IN $\mathbb{Z}_p$-EXTENSIONS

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Abstract. Let k be a real abelian field such that the conductor of every nontrivial character belonging to k agrees with the conductor of k. Note that real quadratic fields satisfy this condition. For a prime p, let k_∞ be the $\mathbb{Z}_p$-extension of k.

The aim of this paper is to produce a set of generators of the Tate cohomology group $\hat{H}^{-1}$ of the circular units of k_n, the nth layer of the $\mathbb{Z}_p$-extension of k, where p is an odd prime. This result generalizes some earlier works which treated the case when k is real quadratic field and used them to study λ-invariants of k.

1. Introduction

Let k be a real abelian field of conductor d. It is known that k admits a unique $\mathbb{Z}_p$-extension for each prime p, namely the basic $\mathbb{Z}_p$-extension $k_\infty = \bigcup_{n\geq 0} k_n$, where $k_n = k\mathbb{Q}_n$. Here $\mathbb{Q}_n$ is the subfield of $\mathbb{Q}(\zeta_{p^{n+1}})$ whose degree over $\mathbb{Q}$ is p^n when p is odd. When $p = 2$, one can define $\mathbb{Q}_n$ similarly with a minor modification. But we will restrict our attention only to odd primes to avoid complexity.

For each $n \geq 0$, let A_n be the Sylow p-subgroup of the ideal class group of k_n. Then it is well known([5],[10]) that there exist integers $\lambda \geq 0$ and μ such that ${}^{\#}A_n = p^{\lambda n+\nu}$ for $n \gg 0$. These constants λ and μ are called Iwasawa invariants. One of the main topics in Iwasawa theory is to investigate these invariants. For instance, Greenberg conjectured([4]) that $\lambda = 0$ and gave several examples supporting the conjecture. Recently, many more affirmative results on the conjecture are given in [2],[3] and [7] when k is real quadratic. In all of these works,

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analysis of cyclotomic units(circular units, in general) is essential because they are related with the class number via the index theorem of Sinnott([9]). In [7], for example, some new information on the λ-invariant of a real quadratic field k was obtained from a particular circular unit of k which generates a certain cohomology group.

The aim of this paper is to find generators of cohomology groups for a wider class of real abelian fields including real quadratic fields so that they could be used, hopefully, in studying λ-invariants. To be precise, let $G_n = Gal(k_n/k_0)$ and C_n be the group of circular units of k_n. Recently, it is proved([8]) that the Tate cohomology group $\hat{H}^{-1}(G_n, C_n)$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^l$, where l is the number of prime ideals of k above p. In this paper, we will find explicit generators of $\hat{H}^{-1}(G_n, C_n)$ when the field k satisfies the following assumption : For each divisor f of the conductor d of k, $k \cap \mathbb{Q}(\zeta_f)$ is either k(this happens when $d = f$) or $\mathbb{Q}$. Note that above assumption is equivalent to saying that every nontrivial character belonging to k is of conductor d. Also note that this assumption is clearly satisfied when k is a real quadratic field or when the conductor d is a prime q. Slightly more generally, the assumption is satisfied when $[k : \mathbb{Q}]$ is a prime. Thus the results of this paper generalize those in [6], where explicit generators of $\hat{H}^{-1}(G_n, C_n)$ are given when $d = q$ is a prime.

This paper is organized as follows. Next section is preliminary. We will review circular units defined by Sinnott briefly and set up notations for the subsequent section. In section 3, we introduce a theorem of Ennola on relations of cyclotomic units and prove several lemmas. And, finally, we exhibit explicit generators.

2. Preliminaries and notations

In this section, we briefly review the definitions of cyclotomic units and circular units and set up notations.

For each integer $s > 0$, we choose a primitive sth root of 1 so that $\zeta_s^{\frac{s}{t}} = \zeta_t$ if $t \mid s$. Let $n \not\equiv 2 \ mod\ 4$ and V be the multiplicative subgroup of $\mathbb{Q}(\zeta_n)^\times$ generated by $\{\pm\zeta_n, 1-\zeta_n^a \mid 1 \le a \le n-1\}$. Define the group C of cyclotomic units of $\mathbb{Q}(\zeta_n)$ by $C = E \cap V$, where E is the group of units of $\mathbb{Q}(\zeta_n)$. In general, for an abelian field M, Sinnott([10]) defines the group of circular units of M as follows. For $n > 2$, let

$$M_n = M \cap \mathbb{Q}(\zeta_n) \ \ and \ \ C_{M_n} = N_{\mathbb{Q}(\zeta_n)/M_n}(C_{\mathbb{Q}(\zeta_n)}),$$

where $C_{\mathbb{Q}(\zeta_n)}$ is the group of cyclotomic units of $\mathbb{Q}(\zeta_n)$. Then the group C_M of circular units of M is defined to be the multiplicative subgroup of $M^\times$ generated by C_{M_n} for all $n > 2$ together with -1. Note that in this definition, we do not have to consider infinitely many n's. Let d be the conductor of M. If $(n, d) = 1$, then $M_n = M \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$. Thus $N_{\mathbb{Q}(\zeta_n)/M_n}(C_{\mathbb{Q}(\zeta_n)}) = \{1\}$. If $d \mid n$, then $M \subset \mathbb{Q}(\zeta_d) \subset \mathbb{Q}(\zeta_n)$. So $N_{\mathbb{Q}(\zeta_n)/M}(C_{\mathbb{Q}(\zeta_n)}) \subset N_{\mathbb{Q}(\zeta_d)/M}(C_{\mathbb{Q}(\zeta_d)})$. Therefore in the definition of circular units it is enough to consider those n's which divide the conductor of n.

As before, k is a real abelian field of conductor d. Let $K = \mathbb{Q}(\zeta_d)$, $F = \mathbb{Q}(\zeta_p)$ and $K' = \mathbb{Q}(\zeta_{pd})$. We denote their cyclotomic $\mathbb{Z}_p$-extension by K_∞,F_∞ and K'_∞, and their nth layers by K_n, F_n and K'_n respectively. Let σ be the topological generator of the Galois group $\Gamma = Gal(K'_\infty/K')$ which maps ζ_{p^n} to $\zeta_{p^n}^{1+p}$ for all $n \geq 1$. Restrictions of σ to various subfields are also denoted by σ. Let $k_{(p)}$ be the decomposition subfield of k for p and let $\Delta = Gal(K/k)$, $\bar{\Delta} = Gal(K/\mathbb{Q})$, $\Delta_p = Gal(K/k_{(p)})$, $\Delta_k = Gal(k/\mathbb{Q})$ and $\Delta_{k,p} = Gal(k_{(p)}/\mathbb{Q})$. Let $[k : \mathbb{Q}] = m$ and $[k_{(p)} : \mathbb{Q}] = l$, so there are l prime ideals in k above p. Elements of Δ, $\bar{\Delta}$ or Δ_p will be denoted by τ's, and those of Δ_k and $\Delta_{k,p}$ by ρ's. The Frobenius automorphism of K for p or its restriction to k is denoted by τ_p. Let R be the set of all roots of 1 in the ring of the p-adic integers, i.e., $R = \{\omega \in \mathbb{Z}_p \mid \omega^{p-1} = 1\}$. Then R can be regarded as the Galois group $Gal(F/\mathbb{Q})$ or any Galois group isomorphic to it such as $Gal(F_n/\mathbb{Q}_n)$. For $m > n$, let $G_{m,n}$ be the Galois group $Gal(K'_m/K'_n)$ and $N_{m,n}$ the norm map $N_{K'_m/K'_n}$ from K'_m to K'_n. We will abbreviate $G_{m,0}$ and $N_{m,0}$ by G_m and N_m, respectively. $G_{m,n}$ will also mean the Galois groups $Gal(k_m/k_n)$, $Gal(F_m/F_n)$ and $Gal(\mathbb{Q}_m/\mathbb{Q}_n)$. Similarly, $N_{m,n}$ will have various meanings. Finally we fix a generator ψ_n of the character group of $Gal(\mathbb{Q}_n/\mathbb{Q})$ such that $\psi_n(\sigma) = \zeta_{p^n}$.

3. Main results

We begin this section with a theorem of Ennola on relations of cyclotomic units which plays a crucial role in our study.

Theorem. *(V. Ennola[1]). Suppose* $\delta = \prod_{1\le a<n}(1-\zeta_n^a)^{x_a}$ *is a root of 1 for some integers* x_a*. Then for every even character* $\chi \neq 1$ *of conductor* f *belonging to* $\mathbb{Q}(\zeta_n)$, $Y(\chi,\delta)=0$, *where*

$$Y(\chi,\delta) = \sum_{\substack{d \\ f|d|n}} \frac{1}{\phi(d)} T(\chi,d,\delta) \prod_{p|d}(1-\overline{\chi}(p)),$$

$$\text{and} \quad T(\chi,d,\delta) = \sum_{\substack{a=1 \\ (a,d)=1}}^{d-1} \chi(a) x_{\frac{n}{d}a}.$$

Lemma 1. *Let* $\chi \neq 1$ *be an even character belonging to* $\mathbb{Q}(\zeta_n)$ *and* δ_1,δ_2,δ *be cyclotomic units in* $\mathbb{Q}(\zeta_n)$*. Then*

(i) $Y(\chi,\delta_1\delta_2) = Y(\chi,\delta_1)+Y(\chi,\delta_2)$.

(ii) If $(\text{root of } 1)\times\delta_1 = (\text{root of } 1)\times\delta_2$, *then* $Y(\chi,\delta_1)=Y(\chi,\delta_2)$.

(iii) For any $\sigma \in Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q}), Y(\chi,\delta^\sigma) = \chi(\sigma)Y(\chi,\delta)$.

(iv) $Y(\chi,\delta^{\sigma-1}) = (\chi(\sigma)-1)Y(\chi,\delta)$.

The proof of Lemma 1 is immediate from the definition of Y.

Next, in the following lemma, we compute $Y(\chi,\delta)$ for various cyclotomic units δ for an even character $\chi=\psi_n\gamma$, where γ is a character of conductor d.

Lemma 2. *Let* $\chi=\psi_n\gamma$ *be an even character of conductor* $p^{n+1}d$*. Then*

(i) $Y(\chi,\zeta_{p^{n+1}}^{\sigma^i\omega^j}-\zeta_d^k) = \frac{1}{\phi(p^{n+1}d)}\psi_n(d\sigma^i)\gamma(kp^{n+1})$

(ii) $Y(\chi,\prod_{i,j,k}(\zeta_{p^{n+1}}^{\sigma^i\omega^j}-\zeta_d^k)^{c_{i,j,k}(\sigma-1)}) = \frac{(\psi_n(\sigma)-1)}{\phi(p^{n+1}d)}\psi_n(d)\gamma(p^{n+1})\alpha(\gamma)$ *for some algebraic integer* $\alpha(\gamma)$ *depending on* γ, *where* $c_{i,j,k}$*'s are integers.*

Proof. Note that $\zeta_{p^{n+1}}^{\sigma^i\omega^j}-\zeta_d^k = \zeta_d^k(\zeta_{p^{n+1}}^{\sigma^i\omega^j}\zeta_d^{-k}-1) = \zeta_d^k(\zeta_{p^{n+1}d}^{\sigma^i\omega^j d-kp^{n+1}}-1)$.

Also note that $\overline{\chi}(q)=0$ for each $q|p^{n+1}d$ since χ is of conductor $p^{n+1}d$.

Thus

$$\begin{aligned} Y(\chi, \zeta_{p^{n+1}}^{\sigma^i\omega^j} - \zeta_d^k) &= \frac{1}{\phi(p^{n+1}d)} T(\chi, p^{n+1}d, 1 - \zeta_{p^{n+1}d}^{\sigma^i\omega^j d - kp^{n+1}}) \\ &= \frac{1}{\phi(p^{n+1}d)} \psi_n \gamma(\sigma^i\omega^j d - kp^{n+1}) \\ &= \frac{1}{\phi(p^{n+1}d)} \psi_n(d\sigma^i)\gamma(kp^{n+1}). \end{aligned}$$

This proves (i).

For (ii), we use Lemma 1. From Lemma 1, we have

$$\begin{aligned} &Y(\chi, \prod_{i,j,k} (\zeta_{p^{n+1}}^{\sigma^i\omega^j} - \zeta_d^k)^{c_{i,j,k}(\sigma-1)}) \\ &= (\psi_n(\sigma) - 1) \sum_{i,j,k} c_{i,j,k} Y(\chi, \zeta_{p^{n+1}}^{\sigma^i\omega^j} - \zeta_d^k) \\ &= (\psi_n(\sigma) - 1) \sum_{i,j,k} c_{i,j,k} \frac{1}{\phi(p^{n+1}d)} \psi_n(d\sigma^i)\gamma(kp^{n+1}) \\ &= \frac{(\psi_n(\sigma) - 1)}{\phi(p^{n+1}d)} \psi_n(d)\gamma(p^{n+1}) \sum_{i,j,k} c_{i,j,k}\psi_n(\sigma^i)\gamma(k) \\ &= \frac{(\psi_n(\sigma) - 1)}{\phi(p^{n+1}d)} \psi_n(d)\gamma(p^{n+1})\alpha(\gamma), \end{aligned}$$

where $\alpha(\gamma) = \sum_{i,j,k} c_{i,j,k}\psi_n(\sigma^i)\gamma(k)$. □

LEMMA 3. *Let* $\delta_n = \prod_{\omega\in R, \tau\in\Delta_p} (\zeta_{p^{n+1}}^{\omega} - \zeta_d^{\tau})$. *Then*

(i) $N_{n,n-1}(\delta_n) = \delta_{n-1}$

(ii) $N_n(\delta_n) = 1$.

Proof. Note that $1 - X^p = \prod_{0\le i\le p-1}(\zeta_p^i - X)$. By putting $X = \zeta_d^{\tau}\zeta_{p^{n+1}}^{-\omega}$, we obtain

$$\prod_{0\le i\le p-1} (\zeta_p^i - \zeta_d^{\tau}\zeta_{p^{n+1}}^{-\omega}) = 1 - \zeta_d^{p\tau}\zeta_{p^n}^{-\omega}.$$

Hence

$$\begin{aligned} N_{n,n-1}(\zeta_{p^{n+1}}^{\omega} - \zeta_d^{\tau}) &= \prod_{0\le i\le p-1} (\zeta_{p^{n+1}}^{(1+ip^n)\omega} - \zeta_d^{\tau}) \\ &= \prod_{0\le i\le p-1} (\zeta_{p^{n+1}}^{\omega}\zeta_p^{i\omega} - \zeta_d^{\tau}) \\ &= \zeta_{p^n}^{\omega} \prod_{0\le i\le p-1} (\zeta_p^{i} - \zeta_{p^{n+1}}^{-\omega}\zeta_d^{\tau}) \\ &= \zeta_{p^n}^{\omega}(1 - \zeta_d^{p\tau}\zeta_{p^n}^{-\omega}) \\ &= \zeta_{p^n}^{\omega} - \zeta_d^{p\tau}. \end{aligned}$$

Therefore

$$\begin{aligned} N_{n,n-1}(\delta_n) &= \prod_{\omega\in R,\tau\in\Delta_p} N_{n,n-1}(\zeta_{p^{n+1}}^{\omega} - \zeta_d^{\tau}) \\ &= \prod_{\omega\in R,\tau\in\Delta_p} (\zeta_{p^n}^{\omega} - \zeta_d^{p\tau}) \\ &= \delta_{n-1}, \end{aligned}$$

since p permutes Δ_p.

Similarly,

$$N_n(\delta_n) = \prod_{\omega\in R,\tau\in\Delta_p} (\zeta_p^{\omega} - \zeta_d^{\tau}).$$

Put $X = \zeta_d^{\tau}$ in the identity $1 - X^p = \prod_{0\le i\le p-1}(\zeta_p^i - X)$ to obtain

$$\frac{1-\zeta_d^{\tau p}}{1-\zeta_d^{\tau}} = \prod_{\omega\in R}(\zeta_p^{\omega} - \zeta_d^{\tau}).$$

Then

$$N_n(\delta_n) = \prod_{\tau\in\Delta_p} \frac{1-\zeta_d^{\tau p}}{1-\zeta_d^{\tau}} = 1.$$

Therefore we get Lemma 3. □

Let $\{\rho, \ldots, \rho_{l-1}, \rho_l = id\}$ be a set of coset representatives of $\bar{\Delta}/\Delta_p$. For each $1 \le i \le l-1$, let

$$\delta_{n,i} = \prod_{\substack{\omega \in R \\ \tau \in \Delta_p}} (\zeta_{p^{n+1}}^{\omega} - \zeta_d^{\tau\rho_i}) \quad and \quad \pi_n = \prod_{\omega \in R} (\zeta_{p^{n+1}}^{\omega} - 1).$$

Then $N_{n,n-1}(\delta_{n,i}) = \delta_{n-1,i}$ and $N_n(\delta_{n,i}) = 1$ by Lemma 3. Also, obviously we have $N_n(\pi_n^{\sigma-1}) = 1$. Thus we have l circular units of k_n whose norm to k equal 1. The next theorem confirms that there are indeed a set of generators of $\hat{H}^{-1}(G_n, C_n)$.

THEOREM. *Suppose that $k \cap \mathbb{Q}(\zeta_f) = k$ or $\mathbb{Q}$ for every f, $f \mid d$. Then $\hat{H}^{-1}(G_n, C_n)$ is generated by $\{\delta_{n,1}, \cdots, \delta_{n,l-1}, \pi_n^{\sigma-1}\}$.*

The proof of the theorem is almost identical with that of section 5 of [6]. The only difference is that the conductor of k in [6] is a prime q, while the conductor d, here, is arbitrary. Although the computations for the general case are more complicated previous lemmas and the next lemma take care of all the difficulties in generalizing the proof of [6] to arbitrary conductor. So we omit the proof.

LEMMA 4. *Suppose that $k \cap \mathbb{Q}(\zeta_f) = k$ or $\mathbb{Q}$ for every f, $f \mid d$. Let u be a circular unit in k_n. Then u can be written as $u = u_0 u_1 u_2 \cdots u_n$, where $u_0 \in C_0$ and for each $1 \le i \le n$, u_i is of the form*

$$u_i = \left(\prod_{\substack{\omega \in R \\ \tau \in \Delta}} (\zeta_{p^{i+1}}^{\omega} - \zeta_d^{\tau})^{\sum_{i,\rho_j \in \Delta_k} a_{i,j}\sigma^i \rho_j} \right) \left(\pi_i^{(\sigma-1)\sum_i b_i \sigma^i} \right).$$

Proof. To describe circular units of k_n, it is enough to consider divisors of the conductor of k_n, which is $p^{n+1}d$ by the remark after the definition of circular units. Thus if $f \mid p^{n+1}d$, then $f = p^{i+1}d_1$, for some $i(-1 \le i \le n)$ and $d_1 \mid d$. Since $\mathbb{Q}(\zeta_f) \cap k_n = k_i$ *or* $\mathbb{Q}_n$. Hence u is of the form

$$u = (N_{\mathbb{Q}(\zeta_d)/k}(v_0))(N_{\mathbb{Q}(\zeta_{p^2 d})/k_1}(v_1))(N_{\mathbb{Q}(\zeta_{p^2})/\mathbb{Q}_1}(\omega_1)) \cdots$$

$$\cdots (N_{\mathbb{Q}(\zeta_{p^{n+1}d})/k_n}(v_n))(N_{\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q}_n}(\omega_n)),$$

where $v_0, v_1, \omega_1, \cdots, v_n$ and ω_n are cyclotomic units in $\mathbb{Q}(\zeta_d)$, $\mathbb{Q}(\zeta_{p^2d})$, $\mathbb{Q}(\zeta_{p^2})$, $\cdots$, $\mathbb{Q}(\zeta_{p^{n+1}d})$ and $\mathbb{Q}(\zeta_{p^{n+1}})$, respectively. Thus u is of the desired form. $\square$

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