

ON THE C_1 -CONSTRUCTION

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ABSTRACT. In [4], it is proved that the algebra (S, k^{14}) is not a C_1 -construction if the field is the real number field. In this paper, we will introduce a different proof of the fact that the algebra (S, k^{14}) is not a C_1 -construction.

1. Introduction

Throughout this paper, k will denote the real number field and $M_n(k)$ will denote the set of all $n \times n$ matrices with entries in k . A commutative k -subalgebra R of $M_n(k)$ is a maximal, commutative k -subalgebra of $M_n(k)$ if and only if R satisfies the following condition: If R^* is a commutative k -subalgebra of $M_n(k)$ with $R \subset R^*$, then $R = R^*$. Let $\mathbf{X}$ denote the category whose objects are ordered pairs (G, H) , where G is a finite dimensional, local, commutative k -algebra and H is a finitely generated, faithful G -module. Let $(B, M) \in \mathbf{X}$. The direct sum $B \oplus M$ of the B -modules B and M can be given the structure of a commutative k -algebra by defining multiplication in the following way.

$$(b_1, m_1)(b_2, m_2) = (b_1b_2, m_2b_1 + m_1b_2), b_i \in B, m_i \in M, i = 1, 2.$$

The commutative ring thus defined is called the idealization of M and will be denoted by $B \rtimes M$.

DEFINITION 1.1. Suppose R is a maximal commutative k -subalgebra of $M_n(k)$. We say R is a (B, N) -construction if R is k -algebra isomorphic to $B \rtimes N^\ell$ for some $(B, N) \in \mathbf{X}$ and a positive integer ℓ .

Here, N^ℓ denotes the direct sum of ℓ copies of B -module N .

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REMARK 1.2. The Courter's algebra R in [1] is a (B, N) -construction.

The B -module $B^\ell \oplus N$ is a $B \rtimes N^\ell$ -module with scalar multiplication defined as follows.

$$(b_1, \dots, b_\ell, n)(b, n_1, \dots, n_\ell) = (b_1 b, \dots, b_\ell b, nb + \sum_{i=1}^{\ell} n_i b_i).$$

REMARK 1.3. For $(B, N) \in \mathbf{X}$, it is known that $B^\ell \oplus N$ is a finitely generated, faithful, $B \rtimes N^\ell$ -module.

If $(G, H), (G', H')$ are two objects in $\mathbf{X}$, then a morphism from (G, H) to (G', H') is an ordered pair (σ, τ) , where $\sigma : G \longrightarrow G'$ is a k -algebra homomorphism, $\tau : H \longrightarrow H'$ is a k -vector space homomorphism with $\tau(hg) = \tau(h)\sigma(g)$ for all $h \in H$ and $g \in G$. We will use the notation $(\sigma, \tau) : (G, H) \longrightarrow (G', H')$ to indicate the morphism (σ, τ) from (G, H) to (G', H') . We call a morphism $(\sigma, \tau) : (G, H) \longrightarrow (G', H')$ an isomorphism if σ is a k -algebra isomorphism and τ is a k -vector space isomorphism. In this case we will use the notation $(G, H) \cong_{(\sigma, \tau)} (G', H')$.

DEFINITION 1.4. With the above notations, $(G, H) \in \mathbf{X}$ is a C_1 -construction if $(G, H) \cong_{(\sigma, \tau)} (B \rtimes N^\ell, B^\ell \oplus N)$ for some $(B, N) \in \mathbf{X}$ and a positive integer ℓ .

REMARK 1.5. If R is the Courter's algebra in [1], then (R, k^{14}) is a C_1 -construction.

2. Main results

As we have proved in [4], the algebra (S, k^{14}) is a maximal commutative subalgebra of matrix algebra of size 14 which is not isomorphic to the Courter's algebra. Recall that the element r in the Jacobson radical

(3) The socle of S , $\text{Soc}(S)$ is generated by $E_{13,1}, E_{13,2}, E_{14,1}, E_{14,2}$. Here, $E_{i,j}$ is a matrix of size 14 whose entries are all zero except the (i,j) -th entry that is 1.

We now introduce a different proof of the fact that (S, k^{14}) is not a C_1 -construction under the real number field.

THEOREM 2.3. *Let k be the real number field. Suppose S is the maximal commutative subalgebra in [4]. Then, (S, k^{14}) is not a C_1 -construction.*

Proof. Let $\{\theta_1, \theta_2\}$ be an arbitrary S -module generator. Then, there exist $a_i, b_i \in k, i = 1, \dots, 14$ such that

$$\theta_1 = \sum_{i=1}^{14} a_i \epsilon_i, \quad \theta_2 = \sum_{i=1}^{14} b_i \epsilon_i$$

Here, $\{\epsilon_1, \dots, \epsilon_{14}\}$ is the standard basis of k^{14} . Now, let $I_i = \text{Ann}_S(\theta_i)$ for $i = 1, 2$. If $r \in I_1$, then we can write $r = aI_{14} + r^*$ for some $a \in k$ and $r^* \in J(S)$. Thus, we have

$$\theta_1 a + \theta_1 r^* = 0$$

This implies

$$a_{13}a = 0, a_{14}a = 0$$

Note that $\epsilon_i \in k^{14}J(S)$ for $i = 1, \dots, 12$ and $\dim_k(k^{14}/k^{14}J(S)) = 2$. Thus, $\{a_{13}\epsilon_{13} + a_{14}\epsilon_{14}, b_{13}\epsilon_{13} + b_{14}\epsilon_{14}\}$ is a k -vector space basis of $k^{14}/k^{14}J(S)$. This implies $a_{13} \neq 0$ or $a_{14} \neq 0$ and hence $a = 0$ and so $r = r^*$. Thus, $I_1 \subset J(S)$ and by the similar way, we get $I_2 \subset J(S)$.

Now, we want to show that $I_i \subset \text{Soc}(S)$ for $i = 1, 2$ by considering the following three cases :

(case 1) $a_{13} \neq 0, a_{14} \neq 0$

(case 2) $a_{13} \neq 0, a_{14} = 0$

(case 3) $a_{13} = 0, a_{14} \neq 0$

We now consider the first case.

If $r \in I_1$, then $r \in J(S)$ and hence r can be written as follows:

$$\left(\begin{array}{cccccccccccccccc} O_{2 \times 2} & & & & & & & O_{2 \times 10} & & & & & & O_{2 \times 2} \\ c_1 & O & & & & & & & & & & & & & \\ O & c_1 & & & & & & & & & & & & & \\ c_2 & O & & & & & & & & & & & & & \\ O & c_2 & & & & & & & & & & & & & \\ c_3 & O & & & & & & & & & & & & & \\ O & c_3 & & & & & & O_{10 \times 10} & & & & & & O_{10 \times 2} \\ c_4 & O & & & & & & & & & & & & & \\ O & c_4 & & & & & & & & & & & & & \\ d_1 & d_2 & & & & & & & & & & & & & \\ d_3 & d_4 & & & & & & & & & & & & & \\ e_1 & e_2 & c_1 + d_1 & d_2 & c_2 + d_3 & d_4 & c_3 & O & c_4 & O & c_1 & c_2 & & & \\ e_3 & e_4 & O & c_1 & O & c_2 & d_1 & c_3 + d_2 & d_3 & c_4 + d_4 & c_3 & c_4 & & & O_{2 \times 2} \end{array} \right)$$

Since $\theta_1 r = 0$, we have the following equations :

$$\begin{aligned} a_{13}(c_1 + d_1) &= 0, & a_{13}d_2 + a_{14}c_1 &= 0, & a_{13}(c_2 + d_3) &= 0, \\ a_{13}d_4 + a_{14}c_2 &= 0, & a_{13}c_3 + a_{14}d_1 &= 0, & a_{14}(c_3 + d_2) &= 0, \\ a_{13}c_4 + a_{14}d_3 &= 0, & a_{14}(c_4 + d_4) &= 0, & a_{13}c_1 + a_{14}c_3 &= 0, \\ a_{13}c_2 + a_{14}c_4 &= 0 \end{aligned} \quad (1)$$

From the above equations we have

$$a_{13}^2 c_1 - a_{14}^2 d_1 = 0, \quad a_{13}^2 c_2 - a_{14}^2 d_3 = 0, \quad a_{13}^2 (c_1 + d_1) = 0, \quad a_{13}^2 (c_2 + d_3) = 0$$

Thus, we have

$$(a_{13}^2 + a_{14}^2)d_1 = 0, \quad (a_{13}^2 + a_{14}^2)d_3 = 0$$

and hence $d_1 = 0$ and $d_3 = 0$. Applying the condition $a_{13} \neq 0$ to the equation (1), we have $c_i = 0, d_i = 0$ for $i = 1, 2, 3, 4$. Thus, r is a linear combination of the matrices $E_{13,1}, E_{13,2}, E_{14,1}, E_{14,2}$ and by Lemma 2.2, $r \in Soc(S)$.

We also can show that I_1 is the subset of $Soc(S)$ in the second case by applying exactly the same methods as the first case.

We now assume $a_{13} = 0, a_{14} \neq 0$. Since $a_{13} = 0$, from the equation (1) we get the following equations:

$$\begin{aligned} a_{14}d_1 &= 0, \quad a_{14}c_2 = 0, \quad a_{14}d_3 = 0, \quad a_{14}(c_3 + d_2) = 0, \\ a_{14}c_3 &= 0, \quad a_{14}(c_4 + d_4) = 0, \quad a_{14}c_1 = 0, \quad a_{14}c_4 = 0 \end{aligned}$$

Since $a_{14} \neq 0$, we have $c_i = 0, d_i = 0$ for $i = 1, 2, 3, 4$ and so $r \in Soc(S)$.

Thus, in all of the three cases, we have $I_1 \subset \text{Soc}(S)$. Similarly, we can show $I_2 \subset \text{Soc}(S)$. This implies $I_1 + I_2 \subset \text{Soc}(S)$. Note that $(0) :_S I_i = J(S)$ for $i = 1, 2$ and moreover, by Lemma 2.2, $\text{Soc}(S)$ is a proper subset of $J(S)$. Therefore, the algebra S doesn't satisfy the condition (1) in Theorem 2.1 and we now conclude the algebra (S, k^{14}) is not a C_1 -construction. $\square$

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