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A HOPF BIFURCATION IN AN ACTIVATOR-INHIBITOR SYSTEM DERIVED FROM A VAN DER POL EQUATION

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Abstract. We are concerned with an activator-inhibitor system proposed by Ohta [8]. The purpose of this paper is to study the dynamics of interfaces in an interfacial problem which is reduced from the system in order to examine how this problem is different from an activator-inhibitor system [3, 7].

1. Introduction

Two-component activator-inhibitor system [8] is a model of developmental biology, population ecology and also the appearance of propagating waves in excitable waves ([4, 5, 6]). The system defined for a band shaped domain is

$$(1)\qquad \begin{cases} \varepsilon\sigma u_t - \varepsilon^2 u_{xx} = f(u,v) - < u+v > \\ v_t - Dv_{xx} = g(u,v),\ t>0,\ x\in(-L/2,L/2) \end{cases}$$

where $\varepsilon, \sigma, \tau$ are all positive constant parameters. The nonlinear terms f and g are van der Pol equation, $f(u,v) = -u + \Theta(u) - v$ and $g(u,v) = u - bv$ where b satisfy the bistable condition and $\Theta(u) = 1$ for $u>0$ and -1 for $u<0$. A spatial average of $u+v$ is $< u+v > = \frac{1}{L}\int_{-L/2}^{L/2}(u+v)dx$.

When the interface width tends to zero as $\varepsilon \to 0$, we can extract systematically from u the slow modes (called *bulk variable*) is a smooth function in the entire space. For $\epsilon = 0$, a free boundary problem with

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two interfaces is obtained :

$$(2)\qquad \begin{cases} v_t = v_{xx} - c^2 v + H(x-\zeta)H(\eta - x) - H(x-\eta)H(\zeta - x) \\ \qquad +\frac{1}{L}(\eta-\zeta) - \frac{1}{2},\ \ (x,t) \in \Omega^+(t) \cup \Omega^-(t) \\ v_x(-\frac{L}{2}, t) = 0 = v_x(\frac{L}{2}, t),\ \ t > 0 \\ \zeta'(t) =\ C(v(\zeta(t)), t),\ \ t > 0 \\ \eta'(t) = -\,C(v(\eta(t), t),\ \ t > 0 \end{cases}$$

where $c^2 = b + 1$. The domains are $\Omega^+(t) = \{(x,t) : \zeta(t) < x < \eta(t), t > 0\}$ and $\Omega^-(t) = \{(x,t) : x < \zeta(t), x > \eta(t), t > 0\}$. The spatial average $< u + v >$ is given by $\frac{1}{L}(\eta - \zeta) - \frac{1}{2}$ and $H(\cdot)$ is a Heaviside step function. The velocity function

$$C(r) = \frac{1}{\sigma}\frac{-2r}{\sqrt{(r + L/2)(L/2 - r)}}$$

of interfaces is continuously differentiable defined on $I := (-L/2, L/2)$.

We shall prove the existence of periodic solutions and the bifurcation of the interface problem. In order to do this, the regular setting of (2) is adapted from [3] in the next section. In section 3, the steady states are examined. We shall state the main theorems in section 4 and give the proofs in section 5.

2. Abstract setting for a regularization

Let $G : [-L/2, L/2]^2 \to \mathbb{R}$ be a Green's function of the differential operator $A := -\frac{d^2}{dx^2} + c^2$ satisfying the boundary conditions. Let the domain of A be

$$D(A) = \{v \in H^{2,2}((-L/2, L/2)) \,:\, v_x(-L/2) = v_x(L/2) = 0\}\,.$$

We define a function $g : [-L/2, L/2]^2 \longrightarrow \mathbb{R}$ by

$$\begin{aligned} & g(x, s, \eta) \\ & \quad := A^{-1}(\tfrac{1}{2} - \tfrac{1}{L}(\eta - s) + H(\cdot - s)H(\eta - \cdot) - H(s - \cdot)H(\cdot - \eta))(x) \\ & \quad = -\tfrac{1}{2} - \tfrac{1}{L}(\eta - s) + 2\int_s^\eta G(x, y)\,dy \end{aligned}$$

and $\gamma : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by $\gamma(s, \eta) := g(s, s, \eta)\,,\ \zeta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by $\zeta(s, \eta) := g(\eta, s, \eta)\,.$ If we use a transformation

$$u(t)(x) := v(x, t) - g(x, s(t), \eta(t))$$

then the problem (2) can be written by an abstract evolution equation

$$(3)\qquad \begin{cases} \dfrac{d}{dt}(u,s,\eta) + \widetilde{A}(u,s,\eta) = f(u,s,\eta) \\ (u,s,\eta)(0) = (u(0),s(0),\eta(0)) = (u_0,-s_0,\eta_0) \end{cases}$$

where $\widetilde{A}$ is a 3×3 matrix whose (1,1)-entry is the operator A and all the others are zero. The nonlinear forcing term f is

$$f(u,s,\eta) = \begin{pmatrix} (2G(x,s(t)) - \frac{1}{L})\, C\Big((u(t)(s(t)) + \gamma(s(t),\eta(t))\Big) \\ +(2G(x,\eta(t)) - \frac{1}{L})\, C\Big(u(t)(\eta(t)) + \zeta(s(t),\eta(t))\Big) \\ C\Big(u(t)(s(t)) + \gamma(s(t),\eta(t))\Big) \\ -C\Big(u(t)(\eta(t)) + \zeta(s(t),\eta(t))\Big) \end{pmatrix}.$$

In [3, 9] the authors proved the well posedness of solutions applying the semigroup ([2]) theory using domains of fractional powers $\alpha \in (3/4, 1]$ of A and $\widetilde{A}$. They obtained that f is a continuously differentiable function from $W \cap \widetilde{X}^\alpha$ to $\widetilde{X}$ where

$$\begin{aligned} W = \;& \{(u,s,\eta) \in C^1([-L/2,L/2]) \times \mathbb{R} \times \mathbb{R} \,:\, u(s) + \gamma(s,\eta) \in I, \\ & u(\eta) + \zeta(s,\eta) \in I\} \subset_{\text{open}} C^1([0,1]) \times \mathbb{R} \times \mathbb{R}, \end{aligned}$$

$X^\alpha := D(A^\alpha)$ and $\widetilde{X}^\alpha := D(\widetilde{A}^\alpha) = X^\alpha \times \mathbb{R} \times \mathbb{R}$.

3. Stationary solutions

We shall deal with the linearized eigenvalue problem for (3) which can be obtained at the stationary solutions. The stationary problem, corresponding to (3), is given by

$$\begin{cases} Au^* = (2G(\cdot,s^*) - \frac{1}{L})\, C(u^*(s^*) + \gamma(s^*,\eta^*)) \\ \qquad\quad +(2G(\cdot,\eta^*) - \frac{1}{L})\, C(u^*(\eta^*) + \zeta(s^*,\eta^*)) \\ 0 = C(u^*(s^*) + \gamma(s^*,\eta^*)) \\ 0 = -\,C(u^*(\eta^*) + \zeta(s^*,\eta^*)) \\ u^{*\prime}(-\frac{L}{2}) = 0 = u^{*\prime}(\frac{L}{2}) \end{cases}$$

for $(u^*, s^*, \eta^*) \in D(\widetilde{A}) \cap W$. This system is equivalent to the pair of equations

$$u^* = 0,\ \ C(\gamma(s^*, \eta^*)\,) = 0 \ \text{ and } \ C(\zeta(s^*, \eta^*)\,) = 0. \tag{4}$$

Theorem 3.1. *The stationary problem of (3) has the stationary solutions* $(0, s^*, \eta^*)$ *for all* $\sigma \neq 0$ *with* $\eta^* = -s^*$ *or* $s^* = -L/4$, $\eta^* = L/4$. *The linearization of* f *at* $(0, s^*, \eta^*)$ *is*

$$\begin{aligned} Df(0, s^*, \eta^*)(\hat{u}, \hat{s}, \hat{\eta}) = \ & (2G(\cdot, s^*) - \tfrac{1}{L})\, f_1(0, s^*, \eta^*)(\hat{u}, \hat{s}, \hat{\eta}) \\ & +(2G(\cdot, \eta^*) - \tfrac{1}{L})\, f_2(0, s^*, \eta^*)(\hat{u}, \hat{s}, \hat{\eta}) \end{aligned}$$

where

$$\begin{aligned} f_1(0, s^*, \eta^*)(\hat{u}, \hat{s}, \hat{\eta}) &= \tfrac{4}{\sigma}\Big(\hat{u}(s^*) + \gamma_s(s^*, \eta^*)\hat{s} + \gamma_\eta(s^*, \eta^*)\hat{\eta}\Big) \\ f_2(0, \eta^*, \eta^*)(\hat{u}, \hat{s}, \hat{\eta}) &= \tfrac{4}{\sigma}\Big(\hat{u}(\eta^*) + \zeta_s(s^*, \eta^*)\hat{s} + \zeta_\eta(s^*, \eta^*)\hat{\eta}\Big). \end{aligned}$$

The pair $(0, s^*, \eta^*)$ *corresponds to a unique steady state* (v^*, s^*, η^*) *of (2) for* $\sigma \neq 0$ *with* $v^*(x) = g(x, s^*, \eta^*)$.

Proof : From the system (4), s^* and η^* are solutions of

$$\begin{cases} \gamma(s, \eta) &= -\frac{1}{2} - \frac{1}{L}(\eta - s) + 2\int_s^\eta G(s, y)\, dy = 0 \\ \zeta(s, \eta) &= -\frac{1}{2} - \frac{1}{L}(\eta - s) + 2\int_s^\eta G(\eta, y)\, dy = 0. \end{cases}$$

Subtracting to each other, we have $s^* + \eta^* = 0$ or $\eta^* - s^* = L/2$. Suppose $s^* + \eta^* = 0$ and let $T(s) = -\frac{1}{2} + \frac{2s}{L} - 2\int_{-s}^{s} G(-s, y)\, dy$. Then $T(0) = -1/2 < 0$, $T(-L/2) = -3/2 + 2\cosh L > 0$ and $T'(s) = \frac{2}{L} - \frac{2\cosh(L/2-2s)}{\sinh(L/2)} < 0$ for $-\frac{L}{2} < s < 0$. Therefore, there is a unique solution $s^* \in (-L/2, 0)$ and $\eta^* = -s^*$. For $\eta^* - s^* = L/2$, s^* is a solution of the following equation $Y(s) = \sinh(2s + \frac{L}{2}) - \sinh(L+s)\, \cosh s + \sinh(\frac{L}{2} - s)\, \cosh(\frac{L}{2} + s)$. Since $Y(0) = \sinh\frac{L}{2}(1 - \cosh\frac{L}{2}) < 0$, $Y(-\frac{L}{2}) = -Y(0)$ and $Y'(s) = 2\cosh(2s + \frac{L}{2}) - \cosh(2s + L)\, \cosh L < 0$ for $-\frac{L}{2} < s < 0$. Hence there is a unique solution (s^*, η^*) with $\eta^* - s^* = L/2$ (in this case, $s^* = -L/4$.) □

4. Main theorems for Hopf bifurcations

We now state the definition for the Hopf bifurcation theory.

DEFINITION 4.1. *Under the assumptions of Proposition 3.1, define (for $3/4 < \alpha \leq 1$) the operator $B \in L(\widetilde{X}^\alpha, \widetilde{X})$ as*

$$B := \frac{\sigma}{4}\, Df(0, s^*, \eta^*)\,.$$

We then define $(0, s^*, \eta^*)$ to be a Hopf point *for (3) if there exists an $\epsilon_0 > 0$ and a C^1-curve*

$$(-\epsilon_0 + \tau^*, \tau^* + \epsilon_0) \mapsto (\lambda(\tau), \phi(\tau)) \in \mathbb{C} \times \widetilde{X}_{\mathbf{C}}$$

($Y_{\mathbf{C}}$ denotes the complexification of the real space Y) of eigendata for $-\widetilde{A} + \tau B$ such that

(i) $(-\widetilde{A} + \tau B)(\phi(\tau)) = \lambda(\tau)\phi(\tau), \quad (-\widetilde{A} + \tau B)(\overline{\phi(\tau)}) = \overline{\lambda(\tau)}\,\overline{\phi(\tau)}$;
(ii) $\lambda(\tau^*) = i\beta$ *with* $\beta > 0$;
(iii) $\mathrm{Re}\,(\lambda) \neq 0$ *for all* $\lambda \in \sigma(-\widetilde{A} + \tau^* B) \setminus \{\pm i\beta\}$;
(iv) $\mathrm{Re}\,\lambda'(\tau^*) \neq 0$ *(transversality).*

The linearized eigenvalue problem is

$$-\widetilde{A}(u, s, \eta) + \tau B(u, s, \eta) = \lambda(u, s, \eta)$$

which is equivalent to

(5)
$$\begin{cases} (A+\lambda)u = \tau(2G(\cdot, s^*) - \frac{1}{L})\Big(u(s^*) + \gamma_s(s^*, \eta^*)\, s + \gamma_\eta(s^*, \eta^*)\, \eta\Big) \\ \qquad\qquad + \tau(2G(\cdot, \eta^*) - \frac{1}{L})\Big(u(\eta^*) + \zeta_s(s^*, \eta^*) s + \zeta_\eta(s^*, \eta^*)\eta\Big) \\ \lambda\, s = \tau\Big(u(s^*) + \gamma_s(s^*, \eta^*)\, s + \gamma_\eta(s^*, \eta^*)\, \eta\Big) \\ \lambda\, \eta = -\tau\Big(u(\eta^*) + \zeta_s(s^*, \eta^*)\, s + \zeta_\eta(s^*, \eta^*)\, \eta\Big) \end{cases}$$

with $\tau = \frac{4}{\sigma}$.

We state our main theorem:

THEOREM 4.2. *The problem (3), respectively (2), has stationary solutions (u^*, s^*, η^*) where $u^* = 0, \eta^* = -s^*$ or $u^* = 0\,, s^* = -L/4, \eta^* = L/4$, respectively (v^*, s^*, η^*) for all $\tau > 0$. Then there exists a unique τ^* such that the linearization $-\widetilde{A} + \tau^* B$ has a purely imaginary pair of eigenvalues. The point $(0, s^*, \eta^*, \tau^*)$ is then a Hopf point for (3) and there* *exists a C^0-curve of nontrivial periodic orbits for (3), respectively (2), bifurcating from $(0, s^*, \eta^*, \tau^*)$, respectively $(v^*, s^*, \eta^*, \tau^*)$.*

In order to prove the main theorem we shall show the next three theorems.

THEOREM 4.3. *Suppose that for $\tau^* \in \mathbb{R} \setminus \{0\}$, the operator $-\widetilde{A} + \tau^* B$ has a unique pair $\{\pm i\beta\}$ of purely imaginary eigenvalues. Then $(0, s^*, \eta^*, \tau^*)$ satisfy the condition (i), (ii), (iii) in Definition 4.1.*

THEOREM 4.4. *Under the same condition as in Theorem 4.3, the point $(0, s^*, \eta^*, \tau^*)$ satisfies the transversality condition. Hence this is a Hopf point for (3).*

THEOREM 4.5. *There exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (5) with $\beta > 0$ for a unique critical point $\tau^* > 0$ in order for $(0, s^*, \eta^*, \tau^*)$ to be a Hopf point.*

5. Proofs of theorems

We shall prove Theorem 4.3 as follows:

Proof of Theorem 4.3 : We assume without loss of generality that $\beta > 0$, and ϕ^* is the (normalized) eigenfunction of $-\widetilde{A} + \tau^* B$ with eigenvalue $i\beta$. We have to show that $(\phi^*, i\beta)$ can be extended to a C^1-curve $\tau \mapsto (\phi(\tau), \lambda(\tau))$ of eigendata for $-\widetilde{A} + \tau B$ with $\mathrm{Re}(\lambda'(\tau^*)) \neq 0$.

For this let $(\psi_0, s_0, \eta_0) \in D(A) \times \mathbb{R} \times \mathbb{R}$. First, we see that $s_0 \neq 0$ and $\eta_0 \neq 0$. For otherwise, by (5), $(A + i\beta)\psi_0 = \mu\, i\beta\, (s_0\, G(\cdot, s^*) + \eta_0\, G(\cdot, \eta^*)) = 0$, which is not possible because A is symmetric. Also, if s_0 or η_0 are zero, the problem will be a single free boundary problem. So without loss of generality, let $s_0 = 1$. Then $E(\psi_0, \eta_0, i\beta, \tau^*) = 0$ by (5), where

$E : D(A)_{\mathbf{C}} \times \mathbb{R} \times \mathbb{C} \times \mathbb{R} \longrightarrow X_{\mathbf{C}} \times \mathbb{C}$,

$$E(u, \eta\,, \lambda, \tau) := \begin{pmatrix} (A + \lambda)u - \tau(u(s^*) + \gamma_s + \gamma_\eta\, \eta) \cdot (2G(\cdot, s^*) - \frac{1}{L}) \\ -\tau(u(\eta^*) - \gamma_\eta - \gamma_s\, \eta)(2G(\cdot, \eta^*) - \frac{1}{L}) \\ \\ \lambda - \tau(u(s^*) - \gamma_s + \gamma_\eta \eta) \\ \\ \lambda\eta + \tau(u(\eta^*) - \gamma_\eta - \gamma_s\, \eta) \end{pmatrix}$$

since $\gamma_s := \gamma_s(s^*,\eta^*) = -\zeta_\eta(s^*,\eta^*)$ and $\gamma_\eta := \gamma_\eta(s^*,\eta^*) = -\zeta_s(s^*,\eta^*)$. The equation $E(u,\eta,\lambda,\tau)=0$ is equivalent to λ being an eigenvalue of $-\widetilde{A}+\tau B$ with eigenfunction $(u,1,\eta)$.

We shall here apply the implicit function theorem to E, and therefore have to check that E is C^1 and that

(6)
$$D_{(u,\eta,\lambda)}E(\psi_0,\eta_0,i\beta,\tau^*) \in L(D(A)_{\mathbf{C}}\times\mathbb{C}, X_{\mathbf{C}}\times\mathbb{C}) \text{ is an isomorphism.}$$

It is easy to see that E is C^1 and in addition, the mapping

$$D_{(u,\eta,\lambda)}E(\psi_0,\eta_0,i\beta,\tau^*)(\hat{u},\hat{\eta},\hat{\lambda}) = \begin{pmatrix} (A+i\beta)\hat{u} - \tau^*\left(\hat{u}(s^*)+\gamma_\eta\,\hat{\eta}\right)\left(2G(\cdot,s^*)-\frac{1}{L}\right) \\ -\tau^*\left(\hat{u}(\eta^*)-\gamma_s\,\hat{\eta}\right)\left(2G(\cdot,\eta^*)-\frac{1}{L}\right) \\ \\ \hat{\lambda}-\tau^*\left(\hat{u}(s^*)+\gamma_\eta\hat{\eta}\,\right) \\ \\ \hat{\lambda}\,\eta_0+i\beta\,\hat{\eta}+\tau^*(\hat{u}(\eta^*)-\gamma_s\,\hat{\eta}\,) \end{pmatrix}$$

is a compact perturbation of the mapping

$$(\hat{u},\hat{\eta},\hat{\lambda}) \longmapsto \Big((A+i\beta)\hat{u},\ \hat{\eta},\ \hat{\lambda}\Big)$$

which is invertible. Thus $D_{(u,\eta,\lambda)}E(\psi_0,\eta_0,i\beta,\tau^*)$ is a Fredholm operator of index 0. Therefore in order to verify (6), it suffices to show that the system

$$D_{(u,\eta,\lambda)}E(\psi_0,\eta_0,i\beta,\tau^*)(\hat{u},\hat{\eta},\hat{\lambda})=0$$

which are

$$(7)\qquad \begin{cases} (A+i\beta)\hat{u}+\hat{\lambda}\psi_0 &= \tau^*\left(2G(\cdot,s^*)-\frac{1}{L}\right)(\hat{u}(s^*)+\gamma_\eta\hat{\eta}) \\ &\quad +\tau^*\left(2G(\cdot,\eta^*)-\frac{1}{L}\right)(\hat{u}(\eta^*)-\gamma_s\hat{\eta}) \\ \hat{\lambda} &= \tau^*\left(\hat{u}(s^*)+\gamma_\eta\,\hat{\eta}\right) \\ \hat{\lambda}\eta_0+i\beta\,\hat{\eta} &= -\tau^*\left(\hat{u}(\eta^*)-\gamma_s\,\hat{\eta}\right) \end{cases}$$

necessarily implies that $\hat{u}=0$, $\hat{\eta}=0$ and $\hat{\lambda}=0$. We define $\phi := \psi_0-(2G(\cdot,s^*)-\frac{1}{L})+(2G(\cdot,\eta^*)-\frac{1}{L})\,\eta_0$ then the first equation of (7) is given by

$$(8)\qquad (A+i\beta)\hat{u}+\hat{\lambda}\phi = -i\beta(2G(\cdot,\eta^*)-\tfrac{1}{L})\,\hat{\eta}\,.$$

Since $(v,s,\eta,\lambda)=(\psi_0,1,\eta,i\beta)$ solves (5), ϕ is a solution to the equation

$$(9)\qquad (A+i\,\beta)\phi = -(2\delta_{s^*}-\tfrac{1}{L})+(2\delta_{\eta^*}-\tfrac{1}{L})\,\eta_0$$

and

(10)

$$i\beta = \tau^*\Big(\phi(s^*) + (2G(s^*,s^*) - \tfrac{1}{L}) + (2G(s^*,\eta^*) - \tfrac{1}{L}) + \gamma_s + \gamma_\eta\,\eta_0\Big)$$
$$-i\beta\eta_0 = \tau^*\Big(\psi_0(\eta^*) - \gamma_\eta - \gamma_s\,\eta_0\Big).$$

From this equation, we have

$$\begin{aligned}
&\tau^*\,\mathrm{Im}\Big(\phi(s^*) - \phi(\eta^*)\,\eta_0\Big) = \beta(1+\eta_0^2)\\
&\mathrm{Re}\Big(\phi(s^*) - \phi(\eta^*)\,\eta_0\Big) = -2(1+\eta_0^2)\textstyle\int_{s^*}^{\eta^*} G_x(s^*,y)dy.
\end{aligned} \tag{11}$$

Equation (9) implies that

$$||A^{1/2}\phi||^2 - i\beta||\phi||^2 = -2(\phi(s^*) - \phi(\eta^*)\eta_0\,) + \frac{1-\eta_0}{L}\int_{-L/2}^{L/2}\phi(x)dx$$

and from (11) we obtain

$$\begin{aligned}
||A^{1/2}\phi||^2 &= -2\mathrm{Re}(\phi(s^*) - \phi(\eta^*)\,\eta_0\,) + \tfrac{1-\eta_0}{L}\textstyle\int_{-L/2}^{L/2}\mathrm{Re}\phi(x)dx\\
&= 4(1+\eta_0^2)\textstyle\int_{s^*}^{\eta^*} G_x(s^*,y)dy - \tfrac{(1-\eta_0)^2}{L}\,\tfrac{c^2}{c^4+\beta^2}\\
\beta||\phi||^2 &= \tfrac{2\beta}{\tau^*}(1+\eta_0^2) - \tfrac{(1-\eta_0)^2}{L}\,\tfrac{\beta}{c^4+\beta^2}
\end{aligned} \tag{12}$$

since $\int(A+i\beta)\phi(x)dx = -(1-\eta_0\,)$. We denote that $\int_{-L/2}^{L/2}|\phi|^2 := ||\phi||^2$. Multiplying ϕ by (8) and $\hat{u}$ by (9) and then integrating,

(13)

$$\begin{aligned}
\hat{\lambda}\textstyle\int\phi^2 = \; & -i\,\beta\,\hat{\eta}\;\textstyle\int(2G(x,\eta^*) - \tfrac{1}{L})\,\phi(x)dx + 2(\hat{u}(s^*) - \hat{u}(\eta^*)\,\eta_0\,)\\
& + \tfrac{1-\eta_0}{L}\textstyle\int\hat{u}(x)dx
\end{aligned}$$

From the equations of (7), we obtain $\hat{u}(s^*) - \hat{u}(\eta^*)\,\eta_0 = \frac{1+\eta_0^2}{\tau^*}\,\hat{\lambda} - \psi_0(\eta^*)\,\hat{\eta}\,$, and from (8), $(c^2+i\beta)\int\hat{u}(x)dx = \hat{\lambda}\int\hat{\phi}(x)dx - i\beta\hat{\eta}$. Furthermore, from (9) we have

$$\begin{aligned}
& i\beta\textstyle\int G(x,\eta^*)\phi(x)dx\\
&\quad = -\textstyle\int A\phi(x)\,G(x,\eta^*)dx - 2(G(s^*,\eta^*) - G(\eta^*,\eta^*)\,\eta_0) + \tfrac{1}{L}(1-\eta_0)\\
&\quad = -\psi_0(\eta^*)
\end{aligned}$$

is obtained. Hence the equation (13) implies that

$$\hat{\lambda}\Big(-\frac{2(1+\eta_0^2)}{\tau^*} + \frac{1}{L}\Big(\frac{1-\eta_0}{c^2+i\beta}\Big)^2 + \int\phi^2\Big) = 0.$$

We suppose that $\hat{\lambda} \neq 0$, then

$$\begin{aligned}
\mathrm{Re} \textstyle\int \phi^2 &= \frac{2(1+\eta_0^2)}{\tau^*} - \frac{1}{L}\frac{c^4-\beta^2}{(c^2+\beta^2)^2}(1-\eta_0)^2 \\
\mathrm{Im} \textstyle\int \phi^2 &= \frac{1}{L}\frac{2c^2\beta}{(c^4+\beta^2)^2}(1-\eta_0)^2.
\end{aligned}$$

From the equation (9),

$$\begin{aligned}
\mathrm{Im} \textstyle\int \phi^2 &= -\tfrac{\beta}{c^4+\beta^2}\big(\tfrac{2c^2(1+\eta_0^2)}{\tau^*} - \tfrac{2c^2\beta}{L(c^4+\beta^2)^2}(1-\eta_0)^2 + 4(1+\eta_0^2)\textstyle\int G_x(s^*,y)dy\big) \\
&= \tfrac{1}{L}\tfrac{2c^2\beta}{(c^4+\beta^2)^2}(1-\eta_0)^2
\end{aligned}$$

which implies that

$$(1+\eta_0^2)(\frac{2c^2}{\tau^*} + 4\int G_x(s^*,y)dy) = 0.$$

Since $\int G_x(s^*,y)dy > 0$, this contracts to the assumption $\hat{\lambda} \neq 0$. Hence, let $\hat{\lambda} = 0$ in (8) and multiply $G(\cdot,\eta^*)$ and then integrate, we have

$$\hat{u}(\eta^*) = -i\beta\int G(x,\eta^*)\hat{u}(x)dx - i\beta\hat{\eta}(\int G^2(x,\eta^*)dx - \tfrac{1}{L}). \tag{14}$$

Multiply $\overline{\hat{u}}$ in (8) and integrate,

$$\hat{u}(\eta^*) = \tfrac{i}{2\beta\hat{\eta}}\big(||A^{1/2}\,\hat{u}||^2 - i\beta||\hat{u}||^2\big) + \tfrac{i\beta\,\hat{\eta}}{L}\big(\tfrac{2c^2-i\beta}{2(c^4+\beta^2)} + 1 - \textstyle\int 2G^2(x,\eta^*)dx\big).$$

Compare with (14), then we obtain

$$\mathrm{Re}\,(\hat{u}(\eta^*)) = \frac{1}{2\hat{\eta}}||\hat{u}||^2 + \frac{\beta^2\,\hat{\eta}}{2L(c^4+\beta^2)} = \gamma_s\,\hat{\eta}$$

implies that

$$||\hat{u}||^2 + \Big(\frac{\beta^2}{L(c^4+\beta^2)} - 2\gamma_s\Big)\hat{\eta}^2 = 0.$$

Since $\gamma_s(s^*,\eta^*) < 0$, this equation implies that $\hat{\eta} = 0$ and $\hat{u} = 0$. □

We shall show the stationary solution is a Hopf point.

Proof of Theorem 4.4 : By implicit differentiation

$$E(\psi_0(\tau),\eta(\tau),\lambda(\tau),\tau) = 0,$$

we have

$$D_{(u,\eta,\lambda)}E(\psi_0, i\beta, \tau^*)(\psi_0'(\tau^*), \eta'(\tau), \lambda'(\tau^*))$$
$$= \begin{pmatrix} (2G(\cdot,s^*)-1/L)(\psi_0(s^*)+\gamma_s+\gamma_\eta\,\eta_0) \\ +(2G(\cdot,\eta^*)-1/L)(\psi_0(\eta^*)-\gamma_\eta-\gamma_s\eta_0) \\ \psi_0(s^*)+\gamma_s+\gamma_\eta\,\eta_0 \\ -(\psi_0(\eta^*)-\gamma_\eta-\gamma_s\eta_0) \end{pmatrix}.$$

This means that the function $\tilde{u} := \psi_0'(\tau^*)$, $\tilde{\eta} := \eta'(\tau^*)$ and $\tilde{\lambda} := \lambda'(\tau^*)$ satisfy the equations

$$(15)\qquad \begin{cases} (A+i\beta)\tilde{u}+\tilde{\lambda}\phi = -i\,\beta\,\tilde{\eta}\,\left(2G(\cdot,\eta^*)-1/L\right) \\ \tilde{\lambda}-\tau^*(\tilde{u}(s^*)+\gamma_\eta\,\tilde{\eta}\,) = \psi_0(s^*)+\gamma_s\,+\gamma_\eta\,\eta_0 \\ \tilde{\lambda}\,\eta_0+i\,\beta\,\tilde{\eta}\,+\tau^*(\tilde{u}(\eta^*)-\gamma_s\,\tilde{\eta}) = -(\psi_0(\eta^*)-\gamma_\eta\,-\gamma_s\,\eta_0) \end{cases}$$

where $\phi := \psi_0 - (2G(\cdot,s^*)-\frac{1}{L}) + (2G(\cdot,\eta^*)-\frac{1}{L})\,\eta_0$. The equations (15) and (9) imply that

$$(16)\qquad \begin{aligned} \tilde{u}(s^*) &= -\gamma_\eta\,\tilde{\eta}\,+\tfrac{\tilde{\lambda}}{\tau^*}-\tfrac{i\beta}{\tau^{*2}} \\ \tilde{u}(\eta^*) &= \gamma_s\,\tilde{\eta}\,-\Big(\tfrac{\tilde{\lambda}}{\tau^*}-\tfrac{i\beta}{\tau^{*2}}\Big)\eta_0-\tfrac{i\beta}{\tau^*}\tilde{\eta}. \end{aligned}$$

Multiplying $\tilde{u}$ by (9) and integrating, and then comparing with (15), we obtain

(17)

$$\begin{aligned} &2(\,\tilde{u}(s^*)-\tilde{u}(\eta^*)\,\eta_0\,) \\ &= \tilde{\lambda}\,\textstyle\int\phi^2+i\,\beta\,\tilde{\eta}\,\int(2G(x,\eta^*)-1/L)\,\phi(x)\,dx+\frac{1-\eta_0}{L}\int\tilde{u}(x)dx \\ &= \tilde{\lambda}\,\textstyle\int\phi^2-2\psi_0(\eta^*)\,\tilde{\eta}+\frac{1}{L}\Big(i\,\beta\,\tilde{\eta}\,\int\phi(x)\,dx-(1-\eta_0)\int\tilde{u}(x)dx\Big). \end{aligned}$$

From (16), $\tilde{u}(s^*)-\tilde{u}(\eta^*)\,\eta_0 = \Big(\frac{\tilde{\lambda}}{\tau^*}-\frac{i\beta}{\tau^{*2}}\Big)(1+\eta_0^2)-(\gamma_\eta+\gamma_s\,\eta_0-\frac{i\beta}{\tau^*}\,\eta_0)\,\tilde{\eta}$. Applying (10), then we obtain

$$(18)\qquad \tilde{\lambda}\Big(\int\phi^2-\tfrac{2(1+\eta_0^2)}{\tau^*}+\tfrac{(1-\eta_0)^2}{L}\tfrac{c^4-\beta^2-2c^2\beta\,i}{(c^4+\beta^2)^2}\Big) = -\tfrac{2i\,\beta}{(\tau^*)^2}\,(1+\eta_0^2)$$

which implies that

$$(19)\qquad \mathrm{Re}\tilde{\lambda}(a^2+b^2) = \frac{2\,\beta(1+\eta_0^2)^2}{\tau^*(c^4+\beta^2)}\Big(\frac{2c^2}{\tau^*}+4\int G_x(s^*,y)dy\Big)$$

where a and b are the real and the imaginary part of $\int \phi^2 - \frac{2(1+\eta_0^2)}{\tau^*} + \frac{(1-\eta_0)^2}{L}\frac{c^4-\beta^2}{(c^4+\beta^2)^2}$. Therefore, $\mathrm{Re}\tilde{\lambda} = \mathrm{Re}\lambda'(\tau^*) > 0$. Hence the transversality condition holds for all $\tau^* > 0$. Therefore, by the Hopf-bifurcation theorem in [3], there exists a family of periodic solutions which bifurcates from the stationary solution as τ passes τ^*. □

The existence and uniqueness of the Hopf critical point are shown.

Proof of Theorem 4.5 : We only need to show that the function $(u, \beta, \tau) \mapsto E(u, \eta, i\beta, \tau)$ has a unique zero with $\beta > 0$ and $\tau > 0$. This means solving the system (5) with $\lambda = i\beta$ and $u = v + (2G(\cdot, s^*) - \frac{1}{L}) - (2G(\cdot, \eta^*) - \frac{1}{L})\,\eta$,

(20)

$$\begin{cases} (A + i\beta)v = -(2\delta_{s^*} - \frac{1}{L}) + (2\delta_{\eta^*} - \frac{1}{L})\,\eta \\ i\beta \;= \tau^* \Big(v(s^*) + (2G(\cdot, s^*) - \frac{1}{L}) - (2G(\cdot, \eta^*) - \frac{1}{L})\,\eta \,+ \gamma_s + \gamma_\eta\,\eta\Big) \\ \qquad = \tau^* \Big(v(s^*) + 2\int G_x(s^*, y)dy\Big) \\ -i\beta\,\eta = \tau^* \Big(v(\eta^*) - 2\eta \int G_x(s^*, y)dy\Big). \end{cases}$$

Thus, we have

$$\begin{aligned} \tfrac{i\beta}{\tau^*}\,(1-\eta) \;&= v(s^*) + v(\eta^*) \\ &\quad + \Big((2G(s^*, s^*) - \tfrac{1}{L}) + (2G(s^*, \eta^*) - \tfrac{1}{L}) + \gamma_s - \gamma_\eta\Big)(1-\eta) \end{aligned}$$

and thus

$$\frac{i\beta}{\tau^*} = -2(G_\beta(s^*, s^*) - G(s^*, s^*)) - 2(G_\beta(s^*, \eta^*) - G(s^*, \eta^*)) + (\gamma_s - \gamma_\eta)$$

where G_β is a Green's function of the differential operator $A + i\beta$. The real and imaginary part of this above equation are given by

$$\beta \;= -\tau^* \;(\mathrm{Im}G_\beta(s^*, s^*) + \mathrm{Im}G_\beta(s^*, \eta^*))$$

$$0 = -2\mathrm{Re}(G_\beta(s^*, s^*) + G_\beta(s^*, \eta^*)) + 2(G(s^*, s^*) - G(s^*, \eta^*)) + \gamma_s - \gamma_\eta.$$

We let

$$F(\beta) := -2\mathrm{Re}(G_\beta(s^*, s^*) + G_\beta(s^*, \eta^*)) + 2(G(s^*, s^*) + G(s^*, \eta^*)) + \gamma_s - \gamma_\eta$$

then $F'(\beta) > 0$ and $F(0) = \gamma_s - \gamma_\eta = \frac{2}{L} - \frac{2}{\sinh L}\,(\cosh L + 1) < 0$. The existence of β is guaranteed since $\lim_{\beta\to\infty} F(\beta) = \frac{2}{L} - \frac{2}{\sinh(L/2)}\,\sinh(L/2 - s^*)\sinh s^* > 0$. □

References

[1] P. Fife, *Dynamics of internal layers and diffusive interfaces*, CMBS-NSF Regional Conference Series in Applied Mathematics, **53**, Philadelphia: SIAM, 1988.

[2] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics **840**, Springer, New York-Berlin, 1981.

[3] Y.M.Ham-Lee, R.Schaaf and R.Thompson, *A Hopf bifurcation in a parabolic free boundary problem*, J. of Comput. Appl. Math. **52** (1994), 305-324.

[4] H. Hodgkin and Huxley, *A qualitative description of membrane current and its application to conduction and excitation in nerves*, J. Physiol **117** (1952), 500–544.

[5] Meinhardt, *Models of Biological Pattern Formulation*, Academic Press, London, 1982.

[6] U. Middya, M.D. Graham, D. Luss, and M. Sheintuch, *Pattern selection in controlled reaction-diffusion systems*, J. Chem. Phys., **98** (1993), 2823–2836.

[7] Y. Nishiura and M. Mimura, *Layer oscillations in reaction-diffusion systems*, SIAM J. Appl. Math. **49** (1989), 481–514.

[8] T. Ohta, M. Mimura, and R. Kobayashi, *Higher dimensional localized patterns in excitable media* , Phys. D **34** (1989), 115 - 144.

[9] Y.M. Ham and S.S.Yum, *The well posedness in a parabolic multiple free boundary problem*, J. of PDE. **8** (1995), 211-218.

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