A HOPF BIFURCATION IN A MULTIPLE FREE BOUNDARY PROBLEM

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Abstract. In this paper we study a Hopf bifurcation in a parabolic multiple free boundary problem. We are dealing with the following problem:

Introduction
Consider the following problem:

\[ v_t = Dv_{xx} - c^2v + H(x - s(t)) - H(x - m(t)) + H(x - n(t)), \]
\[ (x,t) \in \Omega^- \cup \Omega^+, \]
\[ v_x(0,t) = 0 = v_x(1,t), \quad t > 0, \]
\[ v(x,0) = v_0(x), \quad 0 \leq x \leq 1, \]
\[ \tau \frac{ds}{dt} = C(v(s(t),t)), \quad t > 0 \]
\[ \tau \frac{dm}{dt} = -C(v(m(t),t)), \quad t > 0 \]
\[ \tau \frac{dn}{dt} = C(v(n(t),t)), \quad t > 0 \]
\[ s(0) = s_0, \]
\[ m(0) = m_0, \]
\[ n(0) = n_0, \]

where \( v(x,t) \) and \( v_x(x,t) \) are assumed to be continuous in \( \Omega \). (this last requirement imposes a kind of boundary condition at the interface).

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Here $H(y)$ is the Heaviside function, $\Omega = (0, 1) \times (0, \infty)$, $\Omega^- = \{(x, t) \in \Omega : 0 < x < s(t)\}$ and $\Omega^+ = \{(x, t) \in \Omega : x \in (s(t), m(t)) \cup (m(t), 1)\}$.

The velocity of the interface $C(v)$ in (1), which specifies the evolution of the interface $s(t)$, $m(t)$ and $n(t)$, is determined from the first equation in (1) using asymptotic techniques (see in [5,9]). The function $C(v)$ can be calculated as

$$C(v) = \frac{2v - \frac{c_1 - 2a}{c_1 + c_2} \frac{c_1}{c_1 + c_2} - v}{(v + \frac{a}{c_1 + c_2})},$$

where $-c_1 < b < \frac{c_1 (c_2 - a)}{c_1 + a}$ and $c_1, c_2$ are positive constants.

We rewrite (1) as an abstract evolution equation.

$$\frac{d(v, s, m, n)}{dt} + \hat{A}(v, s, m, n) = F(v, s, m, n),$$

$$\text{(v, s, m, n)(0) = (v_0(\cdot), s_0, m_0, n_0)}$$

Here, $\hat{A}$ is a differential operator, and the nonlinear operator $F$.

Since the nonlinear forcing term $F(v, s, m, n)$ contains a Heaviside function in its first component, the combination of this jump discontinuity and the nature of the dependence of $v$ on $s, m$ and $n$ in the second, third and the fourth components of $F$ makes it impossible to find a function space of the form $X = L_p, 1 \leq p \leq \infty$ such that $F$ satisfies a Lipschitz condition on $X \subset X \times R \times R \times R$. Therefore, we need to make a regular problem for this one.

2. Well-Posedness

We now examine a free boundary value problem depending on a new parameter $\mu \in R, \mu = \frac{1}{\tau}$ of the form

$$v_t + Av = H(x - s) - H(x - m) - H(x - n),$$

$$x \in (0, 1) \{s, m, n\}, t > 0$$

$$s(t) = \mu C(v(t), t), \quad t > 0$$

$$m(t) = -\mu C(v(m(t), t)), \quad t > 0$$

$$n(t) = \mu C(v(n(t), t)), \quad t > 0$$

$$v(x, 0) = v_0(x), s(0) = s_0, m(0) = m_0, n(0) = m_0.$$
Here $A$ is the operator $Av = -v_{xx} + c^2v$ together with Neumann boundary conditions $v_x(0) = v_x(1) = 0$. The results in this section apply to any invertible second order operator $A$. On the function $C$, we assume that $C : I \subset \text{open} \to \mathbb{R}$ is continuously differentiable, where $I$ is open in $R$. For the application of semigroup theory to $(F)$, we choose the space $X := L_2((0, 1))$ with norm $|| \cdot ||_2$.

We obtain more regularity for the solution by applying semigroup methods, considering $A$ as a densely defined operator $A : D(A) \subset \text{dense} X \to X$

$$D(A) := \{ v \in H^{2,2}((0, 1)) : v_x(0) = v_x(1) = 0 \}.$$ 

For fixed $s, m$ and $n$, the map $t \mapsto (H(\cdot - s(t)) - H(\cdot - m(t)) - H(\cdot - n(t)))$ is locally Hölder-continuous into $X$ on $(0, T)$, so by standard results for parabolic problems (see e.g. [4]) we obtain from the first equation in $(F)$ that the following regularity holds for $v$.

**Proposition 2.1.** If $(v, s, m, n)$ is a solution of $(F)$, then $v(\cdot, t) \in D(A)$ and the map $t \mapsto v(\cdot, t)$ is in $C^0([0, T), X) \cap C^1((0, T), X)$.

An existence proof for $(F)$ can be obtained along these lines, but it is impossible to get differential dependence on initial conditions this way, because the right hand side $H(\cdot - s) - H(\cdot - m) + H(\cdot - n)$ is not regular enough. The remedy is that we decompose $v$ in $(F)$ into a part $u$, which will be a solution to a regular problem, and a part $g$, which will be explicitly known in terms of Green’s function $G$ of the operator $A$.

**Proposition 2.2.** Let $G : [0, 1]^2 \to \mathbb{R}$ be a Green’s function of the operator $A$. Define $g : [0, 1]^4 \to \mathbb{R}$ by

$$g(x, s, m, n) := \int_s^m G(x, y)dy + \int_m^n G(x, y)dy$$

$$= A^{-1}(H(\cdot - s) - H(\cdot - m) + H(\cdot - n))(x)$$

and for each $i (1 \leq i \leq 3)$ we define $\gamma^i : [0, 1]^3 \to \mathbb{R}$ by

$$\gamma^1(s, m, n) := g(s, s, m, n),$$

$$\gamma^2(s, m, n) := g(m, s, m, n),$$

$$\gamma^3(s, m, n) := g(n, s, m, n).$$
Then \( g(\cdot, s, m, n) \in D(A) \) for all \( s, m, n \) and \( \frac{\partial g}{\partial s}(x, s, m, n) = G(x, s), \frac{\partial g}{\partial m}(x, s, m, n) = G(x, m), \frac{\partial g}{\partial n}(x, s, m, n) = G(x, n) \) are in \( H^{1,\infty}((0, 1) \times (0, 1)) \). Furthermore, \( \gamma^1, \gamma^2 \) and \( \gamma^3 \) are in \( C^\infty([0,1]^3) \).

**Proof.** See [1] □

Applying the well-posedness theorem and the globality theorem together with the starting regularity of solutions to (F) (Proposition 2.2), as well as the regularity of the functions \( g \) and \( \{\gamma^1, \gamma^2, \gamma^3\} \) (Proposition 2.3), we obtain the following result of the global solution.

**THEOREM 2.3.** Let \( S(t) := (s(t), m(t), n(t)) \), and \( S_0 := (s_0, m_0, n_0) \).

Then:

i) For any \( 1 > \alpha > 3/4 \), \( (u_0, S_0) \in W \cap \bar{X}^\alpha \) and \( \mu \in R \), there exists a unique solution \( (u, S)(t) = (u, S)(t; u_0, S_0, \mu) \) of

\[
\begin{align*}
\frac{d}{dt} (u, s, m, n) + \tilde{A}(u, s, m, n) & = \mu f(u, s, m, n) \\
(u, s, m, n)(0) & = (u(0), s(0), m(0), n(0)) = (u_0, s_0, m_0, n_0).
\end{align*}
\]

The solution operator

\[
(u_0, S_0, \mu) \mapsto (u, S)(t; u_0, S_0, \mu)
\]

is continuously differentiable from \( \bar{X}^\alpha \times R \) into \( \bar{X}^\alpha \) for all \( t > 0 \). Then functions \( v(x, t) \) is such that

\[
v(x, t) := u(t)(x) + g(x, S(t))
\]

and satisfies (F) with \( v(\cdot, 0) \in X^\alpha \) and \( v(S_0, 0) \in I \).

ii) If \( (v, S) \) is a solution of (F) for some \( \mu \in R \) with initial conditions \( v_0 \in X^\alpha, 1 > \alpha > 3/4, S_0 \in (0, 1)^3 \) and \( v_0(s_0), v_0(m_0), v_0(n_0) \in I \), then \( (u_0, S_0) := (v_0 - g(\cdot, x_0), S_0) \) in \( \bar{X}^\alpha \cap W \) and

\[
(v(\cdot, t), S(t)) = (u, S)(t; u_0, S_0, \mu) + (g(\cdot, S(t)), 0),
\]

where \( (u, S)(t; u_0, S_0, \mu) \) is the unique solution of (R).

iii) For any \( 1 > \alpha > 3/4 \) and \( \mu \in R \), \( (v_0, S_0) \in U := \{(v, S) \in X^\alpha \times (0, 1) : v(s), v(m), v(n) \in I \} \), the problem (F) has a unique solution for all \( t > 0(v(x, t), S(t)) = (v, S)(x, t; v_0, S_0, \mu) \). Additionally, the mapping \( (v_0, S_0, \mu) \mapsto (v, S)(\cdot, t; v_0, S_0, \mu) \) is continuously differentiable from \( X^\alpha \times R \) into \( X^\alpha \times R \) for all \( t > 0 \).
3. A Hopf bifurcation

The stationary problem for (R) is given by

\begin{align*}
A\mu^* &= \mu C(u^*(s^*) + \gamma(s^*, m^*, n^*)) \cdot G(., s^*) + \mu C(\mu^*(m^*) \\
&\quad + \eta(s^*, m^*, n^*)) \cdot G(., m^*) + \mu C(\mu^*(n^*)) \cdot G(., n^*) \\
0 &= \mu C(u^*(s^*) + \gamma(s^*, m^*, n^*)) \\
0 &= -\mu C(u^*(m^*) + \eta(s^*, m^*, n^*)) \\
0 &= \mu C(u^*(n^*) + \zeta(s^*, m^*, n^*))
\end{align*}

for \((u^*, s^*, m^*, n^*) \in D(\hat{A}) \cap W\). For nonzero \(\mu\) the above system is equivalent to the pair of equations: \(u^* = 0, C(\gamma(s^*, m^*, n^*)) = 0, C(\eta(s^*, m^*, n^*)) = 0\) and \(C(\zeta(s^*, m^*, n^*)) = 0\). We thus obtain the following:

**Proposition 3.1.** If \(0 < \frac{1}{2} - a < \frac{1}{2\sigma}\), then \((R^c)\) has a unique stationary solution for all \(\mu \neq 0\) with \(n^* = 1 - s^* - m^*, s^* \in (0, 1)\). Then linearization of \(f\) at \((0, s^*, m^*, n^*)\) is

\[Df(0, s^*, m^*, n^*)(\hat{u}, \hat{s}, \hat{m}, \hat{n})\]

\[= (\hat{u}(s^*) + \gamma_s(s^*, m^*, n^*)\hat{s} + \gamma_m(s^*, m^*, n^*)m^* \\
+ \gamma_n(s^*, m^*, n^*)n^*) \cdot (G(s^*, n^*, n^*), 1, 0, 0) + (\hat{u}(m^*) + \eta_s(s^*, m^*, n^*)\hat{s} \\
+ \eta_m(s^*, m^*, n^*)m^* + \eta_n(s^*, m^*, n^*)n^*) \cdot (G(s^*, m^*, n^*), 0, -1, 0) \\
+ (\hat{u}(n^*) + \zeta_s(s^*, m^*, n^*)\hat{s} + \zeta_m(s^*, m^*, n^*)m^* \\
+ \zeta_n(s^*, m^*, n^*)n^*) \cdot (G(s^*, m^*, n^*), 0, 0, 1).\]

The pair \((0, s^*, m^*, n^*)\) corresponds to a unique steady state \((v^*, s^*, m^*, n^*)\) of \((F)\) for \(\mu \neq 0\) with \(v^*(x) = g(x, s^*, m^*, n^*)\).

We now show that a Hopf bifurcation occurs as the parameter \(\mu\) approaches zero.

**Theorem 3.2.** (Hopf-Bifurcation) Suppose that \((0, s^*, m^*, n^*, \mu^*)\) is a Hopf point for \((R)\). Then there exist \(\epsilon_1 > 0\) and a \(C^0\)-curve

\[\epsilon \in (-\epsilon_1, \epsilon_1) \mapsto (u_0(\epsilon), s_0(\epsilon), m_0(\epsilon), n_0(\epsilon), p(\epsilon), \mu(\epsilon)) \in \bar{X}^\alpha \times R^+ \times R\]
such that
\[(u, s, m, n)(\cdot; u_0(\epsilon), s_0(\epsilon), m_0(\epsilon), n_0(\epsilon) \mu(\epsilon))\]
is a periodic solution of \((R)\) of period \(p(\epsilon)\).

Moreover, \(u_0(0) = 0, s_0(0) = s^*, m_0(0) = m^*, n_0(0) = n^*, p(0) = 2\pi / \beta \mu(0) = \mu^*\) and
\[
\lim_{\epsilon \to 0} \frac{(u_0(\epsilon), s_0(\epsilon) - s^*, m_0(\epsilon) - m^*, n_0(\epsilon) - n^*)}{\epsilon} = \text{Re}\phi(\mu^*).
\]

**proof.** The proof is similar to the single free boundary case which is in [4]. □

We next have to check \((R)\) for Hopf points. For this, we first solve
the eigenvalue problem

\[-\tilde{A}(u, s, m, n) + \mu B(u, s, m, n) = \lambda(u, s, m, n)\]

which, by Proposition 3.1, is equivalent to

\[(2)\]
\[\begin{align*}
(A + \lambda)\mu &= \mu(\gamma_s(s^*, m^*, n^*)s + \gamma_m(s^*, m^*, n^*)m + \gamma_n(s^*, m^*, n^*)n \\
&\quad + u(s^*)) \cdot G(\cdot, s^*) + \mu(\eta_s(s^*, m^*, n^*)s + \eta_m(s^*, m^*, n^*)m \\
&\quad + \eta_n(s^*, m^*, n^*)n + u(m^*)) \cdot G(\cdot, m^*) + \mu(\zeta_s(s^*, m^*, n^*)s \\
&\quad + \zeta_m(s^*, m^*, n^*)m + \zeta_n(s^*, m^*, n^*)n + u(n^*)) \cdot G(\cdot, n^*)
\end{align*}\]

\[\begin{align*}
\lambda_s &= \mu(\gamma_s(s^*, m^*, n^*)s + \gamma_m(s^*, m^*, n^*)m + \gamma_n(s^*, m^*, n^*)n + u(s^*)) \\
\lambda_m &= -\mu(\eta_s(s^*, m^*, n^*)s + \eta_m(s^*, m^*, n^*)m + \eta_n(s^*, m^*, n^*)n + u(m^*)) \\
\lambda_n &= \mu(\zeta_s(s^*, m^*, n^*)s + \zeta_m(s^*, m^*, n^*)m + \zeta_n(s^*, m^*, n^*)n + u(n^*))
\end{align*}\]

Here we note that
\[
\begin{align*}
\gamma_s(s^*, m^*, n^*) &= -G(s^*, s^*) + \int_{s^*}^{s^*} G_x(s^*, y)dy + \int_{m^*}^{n^*} G(x, s^*)dy \\
&= -\eta_m(s^*, m^*, n^*) = \zeta_n(s^*, m^*, n^*) \\
\gamma_m(s^*, m^*, n^*) &= G(s^*, m^*) = -\eta_n(s^*, m^*, n^*) = \zeta_s(s^*, m^*, n^*)
\end{align*}\]

Furthermore, \(\gamma_s(s^*, m^*, n^*) < 0\) and \(\int_{s^*}^{m^*} G_x(s^*, y)dy = (v^*)(s^*) > 0\).
As a first result, we obtain that it suffices to find a unique, imaginary eigenvalue \(\lambda = i \beta\) of \((2)\) with \(\beta > 0\) for some \(\mu^*\) in order for \((0, s^*, m^*, n^*, \mu^*)\) to be a Hopf point.
Theorem 3.3. Assume that for \( \mu^* \in R - \{0\} \), the \( J \) operator \( -\bar{A} + \mu^* B \) has a unique pair \( \{\pm i\beta\} \) of imaginary eigenvalues. Then \((0, s^*, m^*, n^*, \mu^*)\) is a Hopf point for \( (R) \).

Proof. Without loss of generality, let \( \beta > 0 \), and let \( \phi^* \) be the (normalized) eigenfunction of \( -\bar{A} + \mu^* B \) with eigenvalue \( i\beta \). We have to show that \((\phi^*, i\beta)\) can be extended to a \( C^1 \)-curve \( \mu \mapsto (\phi(\mu), \lambda(\mu)) \) of eigendata for \(-\bar{A} + \mu B\) with \( \lambda(\mu^*) \neq 0 \). To do this let \( \phi^* = (\psi_0, s_0, m_0, n_0) \in D(A) \times R \times R \times R \). First, we can see that \( s_0 \neq 0 \) or \( m_0 \neq 0 \) or \( n_0 \neq 0 \). For, otherwise, by (3), \((A + i\beta)\psi_0 = i\beta \mu_0 \mu^* G(\cdot, s^*) - i\beta \mu_0 \mu^* G(\cdot, m^*) + i\beta \mu_0 \mu^* G(\cdot, n^*) = 0 \) which is impossible, because \( A \) is symmetric. We consider the case of \( s_0 \neq 0, m_0 \neq 0 \) and \( n_0 \neq 0 \). So without loss of generality, let \( s_0 = 1, m_0 = 1 \) and \( n_0 = 1 \).

Then by (3), \( E(\psi_0, i\beta, \mu^*) = 0 \), where \( E : D(A)_C \times C \times R \rightarrow X_C \times C \),

\[
E(u, \lambda, \mu) :=
\begin{pmatrix}
(A + \lambda)u - \mu \cdot (\gamma_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) + u(s^*))G(\cdot, s^*) - \mu \cdot (\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + u(m^*))G(\cdot, m^*) - \mu \cdot (\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + u(n^*))G(\cdot, n^*) \\
\lambda - \mu \cdot (\gamma_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) + u(s^*)) \\
\lambda + \mu \cdot (\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + u(m^*)) \\
\lambda + \mu \cdot (\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + u(n^*))
\end{pmatrix}
\]

The equation \( E(u, \lambda, \mu) = 0 \) is equivalent to saying that \( \lambda \) is an eigenvalue of \(-\bar{A} + \mu B\) with eigenfunction \((u, 1, 1)\). Let’s apply the implicit function theorem to the \( E \). For this, we have to check that \( E \) is in \( C^1 \) and that

(3)
\[
D(u, \lambda)E(\psi_0, i\beta, \mu^*) \in L(D(A)_C \times C, X_C \times C) \text{ is an isomorphism.}
\]

Now it is easy to see that

\[
D_uE(u, \lambda, \mu) \hat{u} = (A + \lambda)\hat{u}(1, 0, 0) - \mu \hat{u}(s^*)G(\cdot, s^*), 1, 0, 0) - \mu \hat{u}(m^*)G(\cdot, m^*), 0, -1, 0) - \mu \hat{u}(n^*)G(\cdot, n^*), 0, 0, 1)
\]
\[ D_\lambda E(u, \lambda, \mu) \lambda = \lambda(u, 1, 1) \]
\[ D_\mu E(u, \lambda, \mu) \mu \]
\[ = -\mu^{}(\gamma_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*))((G(\cdot, s^*), 1, 0, 0)
- \mu^{}(\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*))((G(\cdot, m^*), 0, -1, 0)
- \mu^{}(\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*))((G(\cdot, n^*), 0, 0, 1) \]

so \( E \) is in \( C^1 \). In addition, the mapping

\[ D_{(u, \lambda)} E(\psi_0, i\beta, \mu^*)((\hat{u}, \hat{\lambda}) \]

\[ = \begin{pmatrix}
(A + i\beta)\hat{u} - \mu^*\hat{u}(s^*)G(\cdot, s^*) - \mu^*\hat{u}(m^*)G(\cdot, m^*)
- \mu^*\hat{u}(n^*)G(\cdot, n^*) + \hat{\lambda} \psi_0
\end{pmatrix} \]

is compact perturbation of the mapping

\[ (\hat{u}, \hat{\lambda}) \mapsto ((A + i\beta)\hat{u}, \hat{\lambda}), \]

which is invertible. As a consequence, \( D_{(u, \lambda)} E(\psi_0, i\beta, u^*) \) is a Fredholm operator of index 0. Thus to verify (3), it suffices to show that the system

(4) \[ (A + i\beta)\hat{u} + \hat{\lambda} \psi_0 \]
\[ = \mu^*\hat{u}(s^*)G(\cdot, s^*) + \mu^*\hat{u}(m^*)G(\cdot, m^*) + \mu^*\hat{u}(n^*)G(\cdot, n^*) \]
\[ \hat{\lambda} = \mu^*\hat{u}(s^*) \]
\[ \hat{\lambda} = -\mu^*\hat{u}(m^*) \]
\[ \hat{\lambda} = \mu^*\hat{u}(n^*) \]

necessarily implies that \( \hat{u} = 0 \) and \( \hat{\lambda} = 0 \). Thus let \((\hat{u}, \hat{\lambda})\) be a solution of (4), and define

\[ \psi_1 := \psi_0 - G(\cdot, s^*) + G(\cdot, m^*) - G(\cdot, n^*). \]
Then we have that

\[(A + i\beta)\dot{u} + \lambda\psi_1 = 0.\]

On the other hand, since \(\psi_0\) solve (2) with \(\lambda = i\beta\), \(s = m = n = 1\), we have

\[i\beta G(\cdot, s^*) - i\beta G(\cdot, m^*) + i\beta G(\cdot, n^*) = A\psi_0 + i\beta\psi_0 = (A + i\beta)\psi_1 + \delta_{s^*} - \delta_{m^*} + \delta_{n^*} + i\beta G(\cdot, s^*) - i\beta G(\cdot, m^*) + i\beta G(\cdot, n^*)\]

in the weak sense, where \(\delta_s\) is the delta distribution centered at \(s\). So \(\psi_1\) is a solution of the equations

\[(A + i\beta)\psi_1 = -\delta_{s^*} + \delta_{m^*} - \delta_{n^*}\]

and we have

\[i\beta = \mu^* \cdot (\eta_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) + \psi_0(s^*))
= \mu^* \cdot (\eta_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*)
\quad + \psi_1(s^*) + G(s^*, s^*) - G(s^*, m^*) + G(s^*, n^*))\]

and we have that

\[i\beta = -\mu^* \cdot (\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + \psi_0(s^*))
= -\mu^* \cdot (\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + G(m^*, s^*)
\quad - G(m^*, m^*) + G(m^*, n^*))\]

and we have

\[i\beta = \mu^* \cdot (\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + \psi_0(n^*))
= \mu^* \cdot (\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + \psi_1(n^*) + G(n^*, s^*)
\quad - G(n^*, m^*) + G(n^*, n^*)).\]

Equation (6) implies that

\[-\overline{\psi_1(s^*)} + \overline{\psi_1(m^*)} + \overline{\psi_1(n^*)} = \int_0^1 |A^{\frac{1}{2}}\psi_1|^2 + i\beta \int_0^1 |\psi_1|^2.\]

So we have

\[Im(\psi_1(s^*) - \psi_1(m^*) + \psi_1(n^*)) = \beta \int_0^1 |\psi_1|^2.\]
Adding (7), (8) and (9), we obtain that $\gamma_*(s^*, m^*, n^*), \eta_*(s^*, m^*, n^*), \zeta_*(s^*, m^*, n^*), G(s^*, s^*), G(s^*, m^*)$ and $G(s^*, n^*)$ are real valued. Therefore, since $\beta \neq 0$, we have

$$
(10) \quad \mu^* \int_0^1 |\psi_1|^2 = 2.
$$

From (6), we can calculate $\hat{u}(s^*) - \hat{u}(m^*) + \hat{u}(n^*)$ as

$$
\int_0^1 \psi_1 (A + i\beta) \hat{u} = -\hat{u}(s^*) + \hat{u}(m^*) - \hat{u}(n^*).
$$

Thus together with (5), (6) and (10), we obtain

$$
\hat{\lambda} \int_0^1 \psi_1^2 = \hat{u}(s^*) - \hat{u}(m^*) + \hat{u}(n^*) = 2 \frac{\hat{\lambda}}{\mu^*} = \hat{\lambda} \int_0^1 |\psi_1|^2.
$$

As a result, we have that

$$
\hat{\lambda} \int_0^1 |\psi_1|^2 - \psi_1^2 = 0,
$$

which implies $\hat{\lambda} = 0$. So we conclude that $\hat{\lambda} = 0$ and $\hat{u} = 0$. Therefore we obtain a $C^1$-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata such that $\phi(\mu^*) = \phi^*$ and $\lambda(\mu^*) = i\beta$. It remains to show that $Re(\mu^*) \neq 0$. Let $\phi(\mu) = (\psi(\mu), 1, 1)$. Implicit differentiation of $E(\psi(\mu), \lambda(\mu), \mu) = 0$ implies that

$$
D_{(\mu, \mu)} E(\psi_0, i\beta, \mu^*) (\psi'(\mu^*), \lambda'(\mu^*))
$$

$$
= (\gamma_*(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) + \psi'(\mu^*)(s^*)
\cdot (G(\cdot, s^*), 1, 0, 0) - (\eta_*(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + \psi'(\mu^*)(m^*)
\cdot (G(\cdot, m^*), 0, -1, 0) + (\zeta_*(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + \psi'(\mu^*)(n^*)
\cdot (G(\cdot, n^*), 0, 0, 1).
$$

This means that the function $\hat{u} := \psi'(\mu^*)$ and $\hat{\lambda}(\mu^*)$ satisfy the equations

$$
(11) \quad (A + i\beta) \hat{u} - \mu^* \hat{u}(s^*) G(\cdot, s^*) - \mu^* \hat{u}(m^*) G(\cdot, m^*) - \mu^* \hat{u}(n^*) G(\cdot, n^*) + \hat{\lambda} \psi_0
$$

$$
= - (\gamma_*(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) + \hat{u}(s^*)) G(\cdot, s^*)
- (\eta_*(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + \hat{u}(m^*)) G(\cdot, m^*)
- (\zeta_*(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + \hat{u}(n^*)) G(\cdot, n^*),
$$

\[\square\]
\[ -\mu^* \hat{u}(s^*) + \hat{\lambda} = -\left( \gamma_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) + \hat{u}(s^*) \right) \tag{12} \]

\[ \mu^* \hat{u}(m^*) + \hat{\lambda} = \eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + \hat{u}(m^*) \tag{13} \]

\[ -\mu^* \hat{u}(n^*) + \hat{\lambda} = \zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + \hat{u}(n^*) \tag{14} \]

Using (11), (13), (14) and using \( \psi_1 := \psi_0 - G(\cdot, s^*) + G(\cdot, m^*) - G(\cdot, n^*) \) and (6), we obtain

\[ (A + i\beta)\hat{u} + \hat{\lambda}\psi_1 = 0 \tag{15} \]

and

\[
\begin{align*}
-\bar{u}(s^*) + \bar{u}(m^*) - \bar{u}(n^*) & = \int_0^1 (A + i\beta)\psi_1 \bar{u} \\
& = \int_0^1 \psi_1 \cdot ((A + i\beta)\bar{u} + 2i\beta \hat{u}) \\
& = -\bar{\lambda} \int_0^1 |\psi_1|^2 + 2i\beta \int_0^1 \psi_1 \bar{u} \\
& = -\bar{u} \frac{2}{\mu^*} + 2i\beta \int_0^1 \psi_1 \bar{u}.
\end{align*}
\]

It follows from (12), (13) and (14) that

\[
\begin{align*}
\mu^* (\hat{u}(s^*) - \hat{u}(m^*) + \hat{u}(n^*)) - 2\hat{\lambda} & = 2(\gamma_s + \gamma_m + \gamma_n)(s^*, m^*, n^*) + \hat{u}(s^*) - \hat{u}(m^*) + \hat{u}(n^*),
\end{align*}
\]

where

\[
\begin{align*}
(\gamma_s + \gamma_m + \gamma_n)(s^*, m^*, n^*) & = -G(s^*, s^*) + G(s^*, m^*) - G(s^*, n^*) + \int_{s^*}^{n^*} G_s(s^*, y)dy.
\end{align*}
\]
Thus we obtain

\[ 2i\beta \mu^* \int_0^1 \hat{w} \overline{\psi_1} = 2(\gamma_s + \gamma_m + \gamma_n)(s^*, m^*, n^*) + \lambda \frac{2}{\mu^*} + 2i\beta \int_0^1 \hat{w} \overline{\psi_1}, \]

which implies that

\[ \text{Re} \hat{\lambda} = -\mu^* (\gamma_s + \gamma_m + \gamma_n)(s^*, m^*, n^*) \]

\[ = -\mu^* \frac{1}{csinhc} (1 - coshc(1 - 2s^*)) > 0. \]

Thus the steady state loses stability as \( \mu \) increases beyond \( \mu^* \). For the case \( s_0 \neq 0 \) and \( m_0 = n_0 = 0 \), let \( s_0 = 1 \) and \( m_0 = n_0 = 0 \) in (2). Then by a similar argument, we get the same results with

\[ \text{Re} \hat{\lambda} = -\mu^* \gamma_s (s^*, m^*, n^*) \neq 0. \]

In the case of \( n_0 = s_0 = 0 \) and \( m_0 = 1 \), let \( s_0 = n_0 = 0 \) and \( m_0 = 1 \). Then we have

\[ \text{Re} \hat{\lambda} = -\mu^* \gamma_m (s^*, m^*, n^*) \neq 0. \]

In the case of \( s_0 = m_0 = 0 \), and \( n_0 = 1 \), let \( s_0 = m_0 = 0 \) and \( n_0 = 1 \). Then we have

\[ \text{Re} \hat{\lambda} = -\mu^* \gamma_n (s^*, m^*, n^*) \neq 0. \]

\[ \square \]

Now we will show that, whenever (R) admits a stationary solution, there is a unique \( \mu^* > 0 \) such that \( (0, s^*, m^*, n^*, \mu^*) \) is a Hopf point with \( n^* = 1 - s^* - m^* \). Thus \( \mu^* \) is the origin of a branch of nontrivial periodic orbits. To do this, we only need to show that the function \((u, \beta, \mu) \mapsto E(u, i\beta mu)\) has a unique zero with \( \beta > 0 \) and \( \mu > 0 \). Let \( v' := u - G(\cdot, s^*) - G(\cdot, m^*) - G(\cdot, n^*) \) in the system \( E(u, i\beta, \mu) = 0 \).
Then this system is equivalent to the weak system of equations

\[(A + i\beta)v = -\delta_s^* + \delta_m^* - \delta_n^* \]
\[i\beta = \mu \cdot (\gamma_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) + G(s^*, s^*) - G(s^*, m^*) + G(s^*, n^*) + v(s^*)) \]
\[i\beta = -\mu \cdot (\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + G(m^*, s^*) + G(m^*, m^*) + G(m^*, n^*) + v(m^*)) \]
\[i\beta = \mu \cdot (\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + G(n^*, s^*) - G(n^*, m^*) + G(n^*, n^*) + v(n^*)) \]

Actually, this is just the eigenvalue problem for the formal linearization of \((F)\) about \((v^*, s^*, m^*, n^*)\).

Now the first equation in (16) has for fixed \(\beta \leq 0\), the unique solution \(v = -G_\beta(\cdot, s^*) + G_\beta(\cdot, m^*) - G_\beta(\cdot, n^*)\), where \(G_\beta\) is Green’s function for the operator \(A + i\beta\). Thus it remains to solve the complex valued equation

\[i\beta = \mu \cdot (\gamma_s(s^*, m^*, n^*) + G(s^*, s^*) - G_\beta(s^*, s^*) + G_\beta(s^*, m^*) - G_\beta(s^*, n^*)) \]
\[i\beta = -\mu \cdot (\eta_m(s^*, m^*, n^*) - G(m^*, m^*) - G_\beta(m^*, s^*) + G_\beta(m^*, m^*) - G_\beta(m^*, n^*)) \]
\[i\beta = \mu \cdot (\zeta_n(s^*, m^*, n^*) + G(n^*, n^*) + G_\beta(n^*, s^*) + G_\beta(n^*, m^*) + G_\beta(n^*, n^*)) \]

So we obtain

\[i\beta = \mu \cdot (\gamma_s(s^*, m^*, n^*) + G(s^*, s^*) - G_\beta(s^*, s^*) + G_\beta(s^*, m^*) - G_\beta(s^*, n^*)) \]
\[= \mu \cdot (v^*(s^*)) - G_\beta(s^*, s^*) + G_\beta(s^*, m^*) - G_\beta(s^*, n^*)) \]

where \(\gamma_s(s^*, m^*, n^*) + G(s^*, s^*) = (v^*)(s^*)\). Since \(\gamma_s(s^*, m^*, n^*) + G(s^*, s^*)\) is real valued, that is equivalent to the real valued system

\[\gamma_s(s^*, m^*, n^*) + G(s^*, s^*) - ReG_\beta(s^*, s^*) = 0, \]

\[\mu \cdot Im(G_\beta(s^*, s^*) - G_\beta(s^*, m^*) + G_\beta(s^*, n^*)) + \beta = 0. \]

Since the equation (17) does not depend on \(\mu\), it suffices to find a unique solution \(\beta > 0\) of (17). Using (18), the unique \(\mu^* > 0\) can be calculated when \(\mu^* = Im(G_\beta(s^*, s^*) - G_\beta(s^*, m^*) - G_\beta(s^*, n^*))\) is negative.

The following theorem summarizes what we have proved:
Theorem 3.5. Assume that \( 0 < \frac{1}{2} - a < \frac{1}{c^2} \), so that (R), respectively (2), has a unique stationary solution \((0, s^*, m^*, n^*)\), respectively \((v^*, s^*, m^*, n^*)\), for all \( \mu > 0 \) with \( n^* = 1 - s^* - m^* \). Then there exists a unique \( \mu^* > 0 \) such that the linearization \(-\tilde{A} + \mu^* B\) has a purely imaginary pair of eigenvalues. The point \((0, s^*, m^*, n^*, \mu^*)\) is then a Hopf point for (R) and there exists a \( C^0 \)-curve of non-trivial periodic orbits for (R), (F), respectively, bifurcating from \((0, s^*, m^*, n^*, \mu^*)\), \((v^*, s^*, m^*, n^*, \mu^*)\), respectively.

References


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