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A HOPF BIFURCATION IN A MULTIPLE FREE BOUNDARY PROBLEM

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Abstract. In this paper we study a Hopf bifurcation in a parabolic multiple free boundary problem. We are dealing with the following problem:

Introduction

Consider the following problem:

$$
\begin{aligned}
v_t &= Dv_{xx} - c^2 v + H(x - s(t)) - H(x - m(t)) + H(x - n(t)),\\
&\qquad (x,t) \in \Omega^- \cup \Omega^+,\\
v_x(0,t) &= 0 = v_x(1,t), \qquad t > 0,\\
v(x,0) &= v_0(x), \qquad 0 \le x \le 1,\\
\tau \frac{ds}{dt} &= C(v(s(t),t)), \qquad t > 0\\
\tau \frac{dm}{dt} &= -C(v(m(t),t)), \qquad t > 0\\
\tau \frac{dn}{dt} &= C(v(n(t),t)), \qquad t > 0\\
s(0) &= s_0,\\
m(0) &= m_0,\\
n(0) &= n_0,
\end{aligned} \tag{1}
$$

where $v(x,t)$ and $v_x(x,t)$ are assumed to be continuous in Ω.(this last reuirement imposes a kind of boundary condition at the interface).

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Here $H(y)$ is the Heaviside function, $\Omega = (0,1)\times(0,\infty)$, $\Omega^- = \{(x,t) \in \Omega : 0 < x < s(t)\}$ and $\Omega^+ = \{(x,t) \in \Omega : x \in (s(t), m(t)) \cup (m(t), 1)\}$.

The velocity of the interface $C(v)$ in (1), which specifies the evolution of the interface $s(t)$, $m(t)$ and $n(t)$, is determined from the first equation in (1) using asymptotic techniques (see in [5,9]). The function $C(v)$ can be calculated as

$$C(v) = \frac{2v - \frac{c_1 - 2a}{c_1 + c_2}}{(\frac{c_1 - a}{c_1 + c_2} - v)(v + \frac{a}{c_1 + c_2})},$$

where $-c_1 < b < \frac{c_1(c_2 - a)}{c_1 + a}$ and c_1 ,c_2 are positive constants.

We rewrite (1) as an abstract evolution equation.

$$\frac{d(v,s,m,n)}{dt} + \tilde{A}(v,s,m,n) = F(v,s,m,n),$$
$$(v,s,m,n)(0) = (v_0(.), s_0, m_0, n_0)$$

Here, $\tilde{A}$ is a differential operator, and the nonlinear operator F .

Since the nonlinear forcing term $F(v,s,m,n)$ contains a Heaviside function in its first component, the combination of this jump discontinuity and the nature of the dependence of v on s ,m and n in the second, third and the fourth components of F makes it impossible to find a function space of the form $X = L_p$, $1 \le p \le \infty$ such that F satisfies a Lipschitz condition on $X \subset X \times R \times R \times R$. Therefore, we need to make a regular problem for this one.

2. Well - Posedness

We now examine a free boundary value problem depending on a new parameter $\mu \in R$, $\mu = \frac{1}{\tau}$ of the form

$$\text{(F)}\quad \begin{aligned} & v_t + Av = H(x-s) - H(x-m) - H(x-n), \\ & \qquad\qquad x \in (0,1)\ \{s,m,n\}, t > 0 \\ & s(t) = \mu C(v(t),t)), \qquad t > 0 \\ & m(t) = -\mu C(v(m(t),t)), \qquad t > 0 \\ & n(t) = \mu C(v(n(t),t)), \qquad t > 0 \\ & v(x,0) = v_0(x), s(0) = s_0, m(0) = m_0, n(0) = m_0. \end{aligned}$$

Here A is the operator $Av = -v_{xx} + c^2 v$ together with Neumann boundary conditions $v_x(0) = v_x(1) = 0$. The results in this section apply to any invertible second order operator A. On the function$C$, we assume that $C : I \subset_{open} \rightarrow R$ is continuously differentiable, where I is open in R. For the application of semigroup theory to (F), we choose the space $X := L_2((0,1))$ with norm $||\cdot||_2$.

We obtain more regularity for the solution by applying semigroup methods, considering A as a densely defined operator

$$A : D(A) \subset_{dense} X \rightarrow X$$
$$D(A) := \{v \in H^{2,2}((0,1)) : v_x(0) = v_x(1) = 0\}.$$

For fixed s, m and n , the map $t \mapsto (H(\cdot - s(t)) - H(\cdot - m(t)) - H(\cdot - n(t)))$ is locally Hölder-continuous into X on $(0,T)$, so by standard results for parabolic problems (see e.g. [4]) we obtain from the first equation in (F) that the following regularity holds for v.

Proposition 2.1. *If (v, s, m, n) is a solution of (F), then $v(\cdot,t) \in D(A)$ and the map $t \mapsto v(\cdot,t)$ is in $C^0([0,T), X) \cap C^1((0,T), X)$.*

An existence proof for (F) can be obtained along these lines, but it is impossible to get differential dependence on initial conditions this way, because the right hand side $H(\cdot - s) - H(\cdot - m) + H(\cdot - n)$ is not regular enough. The remedy is that we decompose v in (F) into a part u , which will be a solution to a regular problem, and a part g, which will be explicitly known in terms of Green's function G of the operator A.

Proposition 2.2. *Let $G : [0,1]^2 \rightarrow R$ be a Green's function of the operator A. Define $g : [0,1]^4 \rightarrow R$ by*

$$g(x,s,m,n) := \int_s^m G(x,y)dy + \int_m^n G(x,y)dy$$
$$= A^{-1}(H(\cdot - s) - H(\cdot - m) + H(\cdot - n))(x)$$

and for each $i (1 \leq i \leq 3)$ we define $\gamma^i : [0,1]^3 \rightarrow R$ by

$$\gamma^1(s,m,n) := g(s,s,m,n),$$
$$\gamma^2(s,m,n) := g(m,s,m,n),$$
$$\gamma^3(s,m,n) := g(n,s,m,n).$$

Then $g(\cdot,s,m,n) \in D(A)$ *for all* s,m,n *and* $\frac{\partial g}{\partial s}(x,s,m,n) = G(x,s)$, $\frac{\partial g}{\partial m}(x,s,m,n) = G(x,m)$, $\frac{\partial g}{\partial n}(x,s,m,n) = G(x,n)$ *are in* $H^{1,\infty}((0,1)\times(0,1))$. *Furthermore,* γ^1, γ^2 *and* γ^3 *are in* $C^\infty([0,1]^3)$.

Proof. See [1] □

Applying the well-posedness theorem and the globality theorem together with the starting regularity of solutions to (F) (Proposition 2.2), as well as the regularity of the functions g and $\{\gamma^1,\gamma^2,\gamma^3\}$(Proposition 2.3), we obtain the following result of the global solution.

THEOREM 2.3. *Let* $S(t) := (s(t),m(t),n(t))$, *and* $S_0 := (s_0,m_0,n_0)$. *Then :*
i) For any $1 > \alpha > 3/4$, $(u_0,S_0) \in W \cap \tilde{X}^\alpha$ *and* $\mu \in R$, *there exists a unique solution* $(u,S)(t) = (u,S)(t;u_0,S_0,\mu)$ *of*

(R)
$$\frac{d}{dt}(u,s,m,n) + \tilde{A}(u,s,m,n) = \mu f(u,s,m,n)$$
$$(u,s,m,n)(0) = (u(0),s(0),m(0),n(0)) = (u_0,s_0,m_0,n_0).$$

The solution operator

$$(u_0,S_0,\mu) \mapsto (u,S)(t;u_0,S_0,\mu)$$

is continuously differentiable from $\tilde{X}^\alpha \times R$ *into* $\tilde{X}^\alpha$ *for all* $t > 0$. *Then functions* $v(x,t)$ *is such that*

$$v(x,t) := u(t)(x) + g(x,S(t))$$

and satisfies (F) with $v(\cdot,0) \in X^\alpha$ *and* $v(S_0,0) \in I$.
ii) If (v,S) *is a solution of* (F) *for some* $\mu \in R$ *with initial conditions* $v_0 \in X^\alpha$, $1 > \alpha > 3/4$, $S_0 \in (0,1)^3$ *and* $v_0(s_0), v_0(m_0), v_0(n_0) \in I$, *then* $(u_0,S_0) := (v_0 - g(\cdot,x_0),S_0)$ *in* $\tilde{X}^\alpha \cap W$ *and*

$$(v(\cdot,t),S(t)) = (u,S)(t;u_0,S_0,\mu) + (g(\cdot,S(t)),0),$$

where $(u,S)(t;u_0,S_0,\mu)$ *is the unique solution of* (R).
iii) For any $1 > \alpha > 3/4$ *and* $\mu \in R$, $(v_0,S_0) \in U := \{(v,S) \in X^\alpha \times (0,1) : v(s),v(m),v(n) \in I\}$, *the problem* (F) *has a unique solution for all* $t > 0 (v(x,t),S(t)) = (v,S)(x,t;v_0,S_0,\mu)$. *Additionally, the mapping* $(v_0,S_0,\mu) \mapsto (v,S)(\cdot,t;v_0,S_0,\mu)$ *is continuously differentiable from* $X^\alpha \times R^4$ *into* $X^\alpha \times R^3$ *for all* $t > 0$.

3. A Hopf bifurcation

The stationary problem for (R) is given by

$$\begin{aligned}
&A\mu^* = \mu C(u^*(s^*) + \gamma(s^*, m^*, n^*)) \cdot G(\cdot, s^*) + \mu C(\mu^*(m^*) \\
&\quad + \eta(s^*, m^*, n^*)) \cdot G(\cdot, m^*) + \mu C(\mu^*(n^*)) + \zeta(s^*, m^*, n^*)) \cdot G(\cdot, n^*) \\
&0 = \mu C(u^*(s^*) + \gamma(s^*, m^*, n^*)) \\
&0 = -\mu C(u^*(m^*) + \eta(s^*, m^*, n^*)) \\
&0 = \mu C(u^*(n^*) + \zeta(s^*, m^*, n^*))
\end{aligned}$$

for $(u^*, s^*, m^*, n^*) \in D(\tilde{A}) \cap W$. For nonzero μ the above system is equivalent to the pair of equations: $u^* = 0, C(\gamma(s^*, m^*, n^*)) = 0, C(\eta(s^*, m^*, n^*)` = 0$ and $C(\zeta(s^*, m^*, n^*)) = 0$. We thus obtain the following:

Proposition 3.1. *If $0 < \frac{1}{2} - a < \frac{1}{c^2}$, then $(R`)$ has a unique stationary solution for all $\mu \neq 0$ with $n^* = 1 - s^* - m^*, s^* \in (0,1)$. Then lineariztion of f at $(0, s^*, m^*, n^*)$ is*

$$\begin{aligned}
&Df(0, s^*, m^*, n^*)(\hat{u}, \hat{s}, \hat{m}, \hat{n}) \\
&= (\hat{u}(s^*) + \gamma_s(s^*, m^*, n^*)\hat{s} + \gamma_m(s^*, m^*, n^*)m^* \\
&+ \gamma_n(s^*, m^*, n^*)n^*) \cdot (G(s^*, n^*, n^*), 1, `0, 0) + (\hat{u}(m^*) + \eta_s(s^*, m^*, n^*)\hat{s} \\
&+ \eta_m(s^*, m^*, n^*)m^* + \eta_n(s^*, m^*, n^*)n^*) \cdot (G(s^*, m^*, n^*), 0, -1, 0) \\
&+ (\hat{u}(n^*) + \zeta_s(s^*, m^*, n^*)\hat{s} + \zeta_m(s^*, m^*, n^*)m^* \\
&+ \zeta_n(s^*, m^*, n^*)n^*) \cdot (G(s^*, m^*, n^*), 0, 0, 1).
\end{aligned}$$

The pair $(0, s^*, m^*, n^*)$ corresponds to a unique steady state (v^*, s^*, m^*, n^*) of (F) for $\mu \neq 0$ with $v^*(x) = g(x, s^*, m^*, n^*)$.

We now show that a Hopf bifurcation occurs as the parameter μ approaches zero.

Theorem 3.2. *(Hopf-Bifurcation) Suppose that $(0, s^*, m^*, n^*, \mu^*)$ is a Hopf point for (R). Then there exist $\epsilon_1 > 0$ and a C^0-curve*

$$\epsilon \in (-\epsilon_1, \epsilon_1) \mapsto (u_0(\epsilon), s_0(\epsilon), m_0(\epsilon), n_0(\epsilon), p(\epsilon), \mu(\epsilon)) \in \tilde{X}^\alpha \times R^+ \times R$$

such that

$$(u, s, m, n)(\cdot; u_0(\epsilon), s_0(\epsilon), m_0(\epsilon), n_0(\epsilon)\mu(\epsilon))$$

is a periodic solution of (R) *of period* $p(\epsilon)$.

Moreover, $u_0(0) = 0, s_0(0) = s^*, m_0(0) = m^*, n_0(0) = n^*, p(0) = \frac{2\pi}{\beta}\mu(0) = \mu^*$ *and*

$$lim_{\epsilon \to 0} \frac{(u_0(\epsilon), s_0(\epsilon) - s^*, m_0(\epsilon) - m^*, n_0(\epsilon) - n^*)}{\epsilon} = Re\phi(\mu^*).$$

proof. The proof is similar to the single free boundary case which is in [4]. □

We next have to check (R) for Hopf points. For this, we first solve the eigenvalue problem

$$-\tilde{A}(u, s, m, n) + \mu B(u, s, m, n) = \lambda(u, s, m, n)$$

which, by Proposition 3.1, is equivalent to

(2)

$$\begin{aligned}
&(A + \lambda)\mu = \mu(\gamma_s(s^*, m^*, n^*)s + \gamma_m(s^*, m^*, n^*)m + \gamma_n(s^*, m^*, n^*)n \\
&+ u(s^*)) \cdot G(\cdot, s^*) + \mu(\eta_s(s^*, m^*, n^*)s + \eta_m(s^*, m^*, n^*)m \\
&+ \eta_n(s^*, m^*, n^*)n + u(m^*)) \cdot G(\cdot, m^*) + mu(\zeta_s(s^*, m^*, n^*)s \\
&+ \zeta_m(s^*, m^*, n^*)m + \zeta_n(s^*, m^*, n^*)n + u(n^*)) \cdot G(\cdot, n^*) \\
&\lambda_s = \mu(\gamma_s(s^*, m^*, n^*)s + \gamma_m(s^*, m^*, n^*)m + \gamma_n(s^*, m^*, n^*)n + u(s^*)) \\
&\lambda_m = -\mu(\eta_s(s^*, m^*, n^*)s + \eta_m(s^*, m^*, n^*)m + \eta_n(s^*, m^*, n^*)n + u(m^*)) \\
&\lambda_n = \mu(\zeta_s(s^*, m^*, n^*)s + \zeta_m(s^*, m^*, n^*)m + \zeta_n(s^*, m^*, n^*)n + u(n^*))
\end{aligned}$$

Here we note that

$$\begin{aligned}
\gamma_s(s^*, m^*, n^*) &= -G(s^*, s^*) + \int_{s^*}^{m^*} G_x(s^*, y)dy + \int_{m^*}^{n^*} G_x(s^*, y)dy \\
&= -\eta_m(s^*, m^*, n^*) = \zeta_n(s^*, m^*, n^*) \\
\gamma_m(s^*, m^*, n^*) &= G(s^*, m^*) = -\eta_n(s^*, m^*, n^*) = \zeta_s(s^*, m^*, n^*)
\end{aligned}$$

Furthermore, $\gamma_s(s^*, m^*, n^*) < 0$ and $\int_{s^*}^{m^*} G_x(s^*, y)dy = (v^*)(s^*) > 0$. As a first result, we obtain that it suffices to find a unique, imaginary eigenvalue $\lambda = i\beta$ of (2) with $\beta > 0$ for some μ^* in order for $(0, s^*, m^*, n^*, \mu^*)$ to be a Hopf point.

THEOREM 3.3. *Assume that for $\mu^* \in R - \{0\}$, the] operator $-\tilde{A} + \mu^* B$ has a unique pair $\{^{\pm} i\beta\}$ of imaginary eigenvalues. Then $(0, s^*, m^*, n^*, \mu^*)$ is a Hopf point for (R).*

Proof. Without loss of generality, let $\beta > 0$, and let ϕ^* be the (normalized) eigenfunction of $-\tilde{A} + \mu^* B$ with eigenvalue $i\,\beta$. We have to show that $(\phi^*, i\,\beta)$ can be extended to a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu)$ of eigendata for $-\tilde{A} + \mu B$ with $\lambda(\mu^*) \neq 0$. To do this let $\phi^* = (\psi_0, s_0, m_0, n_0) \in D(A) \times R \times R \times R$. First, we can see that $s_0 \neq 0$ or $m_0 \neq 0$ or $n_0 \neq 0$. For, otherwise, by (3), $(A + i\beta)\psi_0 = i\beta_{s_0} G(\cdot, s^*) - i\beta_{m_0} G(\cdot, m^*)\ + i\beta_{n_0} G(\cdot, n^*) = 0$ which is impossible, because A is symmetric. We consider the case of $s_0 \neq 0, m_0 \neq 0$ and $n_0 \neq 0$. So without loss of generality, let $s_0 = 1$, $m_0 = 1$ and $n_0 = 1$. Then by (3), $E(\psi_0, i\beta, \mu^*) = 0$, where $E : D(A)_C \times C \times R \to X_C \times C$,

$$E(u, \lambda, \mu) :=$$

$$\begin{pmatrix} (A+\lambda)u - \mu\cdot(\gamma_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) \\ +u(s^*))G(\cdot, s^*) - \mu\cdot(\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) \\ +u(m^*))G(\cdot, m^*) - \mu\cdot(\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) \\ +u(n^*))G(\cdot, n^*) \\ \\ \lambda - \mu\cdot(\gamma_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) + u(s^*)) \\ \lambda + \mu\cdot(\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + u(m^*)) \\ \lambda + \mu\cdot(\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + u(n^*)) \end{pmatrix}.$$

The equation $E(u, \lambda, \mu) = 0$ is equivalent to saying that λ is an eigenvalue of $-\tilde{A} + \mu B$ with eigenfunction $(u, 1, 1)$. Let's apply the implicit function theorem to the E. For this, we have to check that E is in C^1 and that

(3)
$$D_{(u,\lambda)} E(\psi_0, i\beta, \mu^*) \in L(D(A)_C \times C, X_C \times C) \ \textit{is an isomorphism.}$$

Now it is easy to see that

$$\begin{aligned} &D_u E(u, \lambda, \mu)\hat{u} \\ &\quad = (A+\lambda)\hat{u}(1,0,0) - \mu\hat{u}(s^*)(G(\cdot, s^*), 1, 0, 0) - \mu\hat{u}(m^*)(G(\cdot, m^*), 0, -1, 0) \\ &\qquad\quad - \mu\hat{u}(n^*)(G(\cdot, n^*), 0, 0, 1) \end{aligned}$$

$$D_\lambda E(u,\lambda,\mu)\hat\lambda = \hat\lambda(u,1,1)$$

$$\begin{aligned} &D_\mu E(u,\lambda,\mu)\hat\mu \\ &\quad= -\hat\mu(\gamma_s(s^*m^*,n^*) + \gamma_m(s^*,m^*,n^*) + \gamma_n(s^*,m^*,n^*)((G(\cdot,s^*),1,0,0) \\ &\quad- \hat\mu(\eta_s(s^*,m^*,n^*) + \eta_m(s^*,m^*,n^*) + \eta_n(s^*,m^*,n^*))(G(\cdot,m^*),0,-1,0) \\ &\quad- \hat\mu(\zeta_s(s^*,m^*,n^*) + \zeta_m(s^*,m^*,n^*) +]\zeta_n(s^*,m^*,n^*))(G(\cdot,n^*),0,0,1) \end{aligned}$$

so E is in C^1. In addition, the mapping

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)(\hat u,\hat\lambda)$$

$$= \begin{pmatrix} (A+i\beta)\hat u - \mu^*\hat u(s^*)G(\cdot,s^*) - \mu^*\hat u(m^*)G(\cdot,m^*) \\ -\mu^*\hat u(n^*)G(\cdot,n^*) + \hat\lambda\psi_0 \\ \\ -\mu^*\hat u(s^*) + \hat\lambda \\ \mu^*\hat u(m^*) + \hat\lambda \\ -\mu^*\hat u(n^*) + \hat\lambda \end{pmatrix}$$

is compact perturbation of the mapping

$$(\hat u,\hat\lambda) \mapsto ((A+i\beta)\hat u, \hat\lambda),$$

which is invertible. As a consequence, $D_{(u,\lambda)}\ E(\psi_0, i\beta, u^*)$ is a Fredholm operator of index 0. Thus to verify (3), it suffices to show that the system

$$\begin{aligned} (4)\qquad &(A+i\beta)\hat u + \hat\lambda\psi_0 \\ &= \mu^*\hat u(s^*)G(\cdot,s^*) + \mu^*\hat u(m^*)G(\cdot,m^*) + \mu^*\hat u(n^*)G(\cdot,n^*) \\ &\hat\lambda = \mu^*\hat u(s^*) \\ &\hat\lambda = -\mu^*\hat u(m^*) \\ &\hat\lambda = \mu^*\hat u(n^*) \end{aligned}$$

necessarily implies that $\hat u = 0$ and $\hat\lambda = 0$. Thus let $(\hat u,\hat\lambda)$ be a solution of (4), and define

$$\psi_1 := \psi_0 - G(\cdot,s^*) + G(\cdot,m^*) - G(\cdot,n^*).$$

Then we have that

$$(A + i\,\beta)\hat{u} + \hat{\lambda}\psi_1 = 0. \tag{5}$$

On the other hand, since ψ_0 solve (2) with $\lambda = i\beta$, $s = m = n = 1$, we have $i\beta G(\cdot, s^*)\ -i\beta G(\cdot, m^*)\ +i\beta G(\cdot, n^*) = A\psi_0 + i\beta\psi_0 = (A + i\beta)\psi_1\ +\delta_{s^*}\ -\delta_{m^*}\ +\delta_{n^*}\ +i\beta G(\cdot, s^*)\ -i\beta G(\cdot, m^*)\ +i\beta G(\cdot, n^*)$ in the weak sense, where δ_s is the delta distribution centered at s. So ψ_1 is a solution of the equations

(6)
$$(A + i\beta)\psi_1 = -\delta_{s^*} + \delta_{m^*} - \delta_{n^*}$$

(7)
$$\begin{aligned} i\beta &= \mu^* \cdot (\gamma_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) + \psi_0(s^*)) \\ &= \mu^* \cdot (\gamma_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^* + \gamma_n(s^*, m^*, n^*) \\ &\qquad + \psi_1(s^*) + G(s^*, s^*) - G(s^*, m^*) + G(s^*, n^*)) \end{aligned}$$

and we have that

(8)
$$\begin{aligned} i\beta &= -\mu^* \cdot (\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + \psi_0(s^*)) \\ &= -\mu^* \cdot (\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + G(m^*, s^*) \\ &\quad - G(m^*, m^*) + G(m^*, n^*)) \end{aligned}$$ $\blacksquare$

and we have

(9)
$$\begin{aligned} i\beta &= \mu^* \cdot (\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + \psi_0(n^*)) \\ &= \mu^* \cdot (\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + \psi_1(n^*) + G(n^*, s^*) \\ &\quad - G(n^*, m^*) + G(n^*, n^*)). \end{aligned}$$ $\blacksquare$

Equation (6) implies that

$$-\overline{\psi_1(s^*)} + \overline{\psi_1(m^*)} + \overline{\psi_1(n^*)} = \int_0^1 |A^{\frac{1}{2}}\psi_1|^2 + i\beta \int_0^1 |\psi_1|^2.$$

So we have

$$Im(\psi_1(s^*) - \psi_1(m^*) + \psi_1(n^*)) = \beta \int_0^1 |\psi_1|^2.$$

Adding (7), (8) and (9), we obtain that $\gamma_s(s^*, m^*, n^*)$, $\eta_s(s^*, m^*, n^*)$, $\zeta_s(s^*, m^*, n^*)$, $G(s^*, s^*)$, $G(s^*, m^*)$ and $G(s^*, n^*)$ are real valued. Therefore, since $\beta \neq 0$, we have

$$\mu^* \int_0^1 |\psi_1|^2 = 2. \tag{10}$$

From (6), we can calculate $\hat{u}(s^*)$ $-\hat{u}(m^*)$ $+\hat{u}(n^*)$ as

$$\int_0^1 \psi_1(A + i\beta)\hat{u} = -\hat{u}(s^*) + \hat{u}(m^*) - \hat{u}(n^*).$$

Thus together with (5), (6) and (10), we obtain

$$\hat{\lambda} \int_0^1 {\psi_1}^2 = \hat{u}(s^*) - \hat{u}(m^*) + \hat{u}(n^*) = 2\frac{\hat{\lambda}}{u^*} = \hat{\lambda} \int_0^1 |\psi_1|^2.$$

As a result, we have that

$$\hat{\lambda}(\int_0^1 |\psi_1|^2 - {\psi_1}^2) = 0,$$

which implies $\hat{\lambda} = 0$. So we conclude that $\hat{\lambda} = 0$ and $\hat{u} = 0$. Therefore we obtain a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata such that $\phi(\mu^*) = \phi^*$ and $\lambda(\mu^*) = i\beta$. It remains to show that $Re\lambda(\mu^*) \neq 0$. Let $\phi(\mu) = (\psi(\mu, 1, 1)$. Implicit differentiation of $E(\psi(\mu), \lambda(\mu), \mu) = 0$ implies that

$$\begin{aligned}
&D_{(u,\mu)}E(\psi_0, i\beta, \mu^*)(\psi'(\mu^*), \lambda'(\mu^*)) \\
&= (\gamma_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) + \psi'(\mu^*)(s^*) \\
&\cdot (G(\cdot, s^*), 1, 0, 0) - (\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + \psi'(\mu^*)(m^*)) \\
&\cdot (G(\cdot, m^*), 0, -1, 0) + (\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + \psi'(\mu^*)(n^*)) \\
&\cdot (G(\cdot, n^*), 0, 0, 1). \qquad \blacksquare
\end{aligned}$$

This means that the function $\hat{u} := \psi'(\mu^*)$ and $\hat{\lambda}(\mu^*)$ satisfy the equations

(11)

$$\begin{aligned}
&(A + i\beta)\hat{u} - \mu^*\hat{u}(s^*)G(\cdot, s^*) - \mu^*\hat{u}(m^*)G(\cdot, m^*) - \mu^*\hat{u}(n^*)G(\cdot, n^*) + \hat{\lambda}\psi_0 \\
&= -(\gamma_s(s^*, m^*, n^*) + \gamma_m(s^*, m^*, n^*) + \gamma_n(s^*, m^*, n^*) + \hat{u}(s^*))G(\cdot, s^*) \\
&- (\eta_s(s^*, m^*, n^*) + \eta_m(s^*, m^*, n^*) + \eta_n(s^*, m^*, n^*) + \hat{u}(m^*))G(\cdot, m^*) \\
&- (\zeta_s(s^*, m^*, n^*) + \zeta_m(s^*, m^*, n^*) + \zeta_n(s^*, m^*, n^*) + \hat{u}(n^*))G(\cdot, n^*), \qquad \blacksquare
\end{aligned}$$

(12)
$$-\mu^*\hat{u}(s^*)+\hat{\lambda} = -(\gamma_s(s^*,m^*,n^*)+\gamma_m(s^*,m^*,n^*)+\gamma_n(s^*,m^*,n^*)+\hat{u}(s^*))$$

(13)
$$\mu^*\hat{u}(m^*)+\hat{\lambda} = \eta_s(s^*,m^*,n^*)+\eta_m(s^*,m^*,n^*)+\eta_n(s^*,m^*,n^*)+\hat{u}(m^*)$$

(14)
$$-\mu^*\hat{u}(n^*)+\hat{\lambda} = \zeta_s(s^*,m^*,n^*)+\zeta_m(s^*,m^*,n^*)+\zeta_n(s^*,m^*,n^*)+\hat{u}(n^*).$$

Using (11), (13), (14) and using $\psi_1 := \psi_0 - G(\cdot,s^*) + G(\cdot,m^*) - G(\cdot,n^*)$ and (6), we obtain

$$(A+i\beta)\hat{u}+\hat{\lambda}\psi_1 = 0 \tag{15}$$

and

$$\begin{aligned}
&-\overline{\hat{u}(s^*)}+\overline{\hat{u}(m^*)}-\overline{\hat{u}(n^*)}\\
&=\int_0^1 (A+i\beta)\psi_1\overline{\hat{u}}\\
&=\int_0^1 \psi_1\cdot(\overline{(A+i\beta)\hat{u}}+2i\beta\hat{u})\\
&=-\overline{\hat{\lambda}}\int_0^1|\psi_1|^2+2i\beta\int_0^1\psi_1\overline{\hat{u}}\\
&=-\overline{\hat{u}}\frac{2}{\mu^*}+2i\beta\int_0^1\psi_1\overline{\hat{u}}.
\end{aligned}$$

It follows from (12), (13) and (14) that

$$\begin{aligned}
&\mu^*(\hat{u}(s^*)-\hat{u}(m^*)+\hat{u}(n^*))-2\hat{\lambda}\\
&\quad=2(\gamma_s+\gamma_m+\gamma_n)(s^*,m^*,n^*)+\hat{u}(s^*)-\hat{u}(m^*)+\hat{u}(n^*),
\end{aligned}$$

where

$$\begin{aligned}
&(\gamma_s+\gamma_m+\gamma_n)(s^*,m^*n^*)\\
&=-G(s^*,s^*)+G(s^*,m^*)-G(s^*,n^*)+\int_{s^*}^{n^*}G_s(s^*,y)dy.
\end{aligned}$$

Thus we obtain

$$2i\beta\mu^* \int_0^1 \hat{u}\overline{\psi_1} = 2(\gamma_s + \gamma_m + \gamma_n)(s^*, m^*, n^*) + \hat{\lambda}\frac{2}{\mu^*} + 2i\beta \int_0^1 \hat{u}\overline{\psi_1},$$

which implies that

$$\begin{aligned} Re\hat{\lambda} &= -\mu^*(\gamma_s + \gamma_m + \gamma_n)(s^*, m^*, n^*) \\ &= -\mu^* \frac{1}{csinhc}(1 - coshc(1 - 2s^*)) > 0. \end{aligned}$$

Thus the steady state loses stability as μ increases beyond μ^*. For the case $s_0 \neq 0$ and $m_0 = n_0 = 0$, let $s_0 = 1$ and $m_0 = n_0 = 0$ in (2). Then by a similar argument, we get the same results with

$$Re\hat{\lambda} = -\mu^*\gamma_s(s^*, m^* n^*) \neq 0.$$

In the case of $n_0 = s_0 = 0$ and $m_0 = 1$, let $s_0 = n_0 = 0$ and $m_0 = 1$. Then we have

$$Re\hat{\lambda} = -\mu^*\gamma_m(s^*, m^*, n^*) \neq 0.$$

In the case of $s_0 = m_0 = 0$, and $n_0 = 1$, let $s_0 = m_0 = 0$ and $n_0 = 1$. Then we have

$$Re\hat{\lambda} = -\mu^*\gamma_n(s^*, m^*, n^*) \neq 0.$$

□

Now we will show that, whenever (R) admits a stationary solution, there is a unique $\mu^* > 0$ such that $(0, s^*, m^*, n^*, \mu^*)$ is a Hopf point with $n^* = 1 - s^* - m^*$. Thus μ^* is the origin of a branch of nontrivial periodic orbits. To do this, we only need to show that the function $(u, \beta, \mu) \mapsto E(u, i\beta mu)$ has a unique zero with $\beta > 0$ and $\mu > 0$. Let $v‘ := u - G(\cdot, s^*)\ -G(\cdot, m^*) - G(\cdot, n^*)$ in the system $E(u, i\beta, \mu) = 0$.

Then this system is equivalent to the weak system of equations

(16)

$$
\begin{aligned}
&(A+i\beta)v = -\delta_s^* + \delta_m{}^* - \delta_n{}^* \\
&i\beta = \mu\cdot(\gamma_s(s^*,m^*,n^*) + \gamma_m(s^*,m^*,n^*) + \gamma_n(s^*,m^*,n^*) + G(s^*,s^*) \\
&\qquad - G(s^*,m^*) + G(s^*,n^*) + v(s^*)) \\
&i\beta = -\mu\cdot(\eta_s(s^*,m^*,n^*) + \eta_m(s^*,m^*,n^*) + \eta_n(s^*,m^*,n^*) + G(m^*,s^*) + G(m^*,m^*) \\
&\qquad + G(m^*,n^*) + v(m^*)) \\
&i\beta = \mu\cdot(\zeta_s(s^*,m^*,n^*) + \zeta_m(s^*,m^*,n^*) + \zeta_n(s^*,m^*,n^*) + G(n^*,s^*) - G(n^*,m^*) \\
&\qquad + G(n^*,n^*) + v(n^*)).
\end{aligned}
$$

Actually, this is just the igenvalue problem for the forrmal linearization of (F) about (v^*,s^*,m^*,n^*).
Now the first equation in (16) has for fixed $\beta \le 0$, the unique solution $v = -G_\beta(\cdot,s^*)$ $+G_\beta(\cdot,m^*)$ $-G_\beta(\cdot,n^*)$, where G_β is Green's function for the operator $A+i\beta$. Thus it remains to solve the complex valued equation

$$
\begin{aligned}
&i\beta = \mu\cdot(\gamma_s(s^*,m^*,n^*) + G(s^*,s^*) - G_\beta(s^*,s^*) + G_\beta(s^*,m^*) - G_\beta(s^*,n^*)) \\
&i\beta = -\mu\cdot(\eta_m(s^*,m^*,n^*) - G(m^*,m^*) - G_\beta(m^*,s^*) + G_\beta(m^*,m^*) - G_\beta(m^*,n^*)) \\
&i\beta = \mu\cdot(\zeta_n(s^*,m^*,n^*) + G(n^*,n^*) + G_\beta(n^*,s^*) + G_\beta(n^*,m^*) + G_\beta(n^*,n^*))\,.
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
i\beta &= \mu\cdot(\gamma_s(s^*,m^*,n^*) + G(s^*,s^*) - G_\beta(s^*,s^*) + G_\beta(s^*,m^*) - G_\beta(s^*,n^*)) \\
&= \mu\cdot((v^*)'(s^*) - G_\beta(s^*,s^*) + G_\beta(s^*,m^*) - G_\beta(s^*,n^*))\,,
\end{aligned}
$$

where $\gamma_s(s^*,m^*,n^*)$ $+G(s^*,s^*) = (v^*)'(s^*)$. Since $\gamma_s(s^*,m^*,n^*)$ $+G(s^*,s^*)$ is real valued, that is equivalent to the real valued system

$$\gamma_s(s^*,m^*,n^*) + G(s^*,s^*) - ReG_\beta(s^*,s^*) = 0, \tag{17}$$

$$\mu\cdot Im(G_\beta(s^*,s^*) - G_\beta(s^*,m^*) + G_\beta(s^*,n^*)) + \beta = 0. \tag{18}$$

Since the equation (17) does not depend on μ, it suffieces to find a unique solution $\beta > 0$ of (17). Using (18), the unique $\mu^* > 0$ can be calculated when $Im(G_\beta(s^*,s^*)$ $-G_\beta(s^*,m^*)$ $-G_\beta(s^*,n^*))$ is negative.

The following theorem summarizes what we have proved :

THEOREM 3.5. *Assume that* $0 < \frac{1}{2} - a < \frac{1}{c^2}$ *,so that (R), respectively (2), has a unique stationary solution* $(0, s^*, m^*, n^*)$*, respectively* (v^*, s^*, m^*, n^*)*, for all* $\mu > 0$ *with* $n^* = 1 - s^* - m^*$*. Then there exists a unique* $\mu^* > 0$ *such that the linearization* $-\tilde{A} + \mu^* B$ *has a purely imaginary pair of eigenvalues. The point* $(0, s^*, m^*, n^*, \mu^*)$ *is then a Hopf point for (R) and there exists a* C^0-*curve of nontrivial periodic orbits for (R), (F), respectively, bifurcating from* $(0, s^*, m^*, n^*, \mu^*), (v^*, s^*, m^*, n^*, \mu^*)$*, respectively.*

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