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EXTREME POINTS RELATED TO MATRIX ALGEBRAS

Tae Keug Lee

Abstract. Let A denote the set $\{a \in M_n | a \geq 0, tr(a) = 1\}$, $St(M_n)$ the set of all states on M_n, and $PS(M_n)$ the set of all pure states on M_n. We show that there are one-to-one correspondences between A and $St(M_n)$, and between the set of all extreme points of A and $PS(M_n)$. We find a necessary and sufficient condition for a state on $M_{n_1} \oplus \cdots \oplus M_{n_k}$ to be extended to a pure state on $M_{n_1+\cdots+n_k}$.

1. Introduction and Preliminaries

The representation theory plays an important role in the operator algebra and it is closely related to states. Since pure states give irreducible representations by the GNS construction, it is natural that the study of pure states is our concern. Let $\mathbb{C}^n$ be the $n-$dimensional vector space over the complex field $\mathbb{C}$ and let $< , >$ denote the standard inner product on $\mathbb{C}^n$. Let M_n be the set of all $n \times n$ complex matrices and I_n the identity matrix of M_n. An $n \times n$ matrix a is called *positive*, denoted $a \geq 0$, if it is hermitian and $< ax, x >$ is non-negative for all $x \in \mathbb{C}^n$. If $f : M_n \to M_m$ is a linear map, then f is called *positive* provided that it maps positive matrices of M_n to positive matrices of M_m. If a linear functional $f : M_n \to \mathbb{C}$ is positive and $f(I_n) = 1$, then f is called a *state* on M_n. We denote the set of all states on M_n by $St(M_n)$. Let K be a subset of a vector space X. An element $a \in K$ is called an *extreme point* of K provided that $x = y = a$ whenever $x, y \in K$, $0 < t < 1$, and $a = tx+(1-t)y$. A state f on M_n is said to be *pure* if every positive linear functional on M_n that is dominated by f is of the form λf $(0 \leq \lambda \leq 1)$. We denote by $PS(M_n)$ the set of all pure states on M_n. It is known that

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the set of all extreme points of $St(M_n)$ is the set of all pure states of M_n. The properties of positive linear maps were studied in [1,2,3,4]. In section 2, we show that there are one-to-one correspondences between the set $\{a \in M_n \big| a \geq 0, tr(a) = 1\}$ and $St(M_n)$ and between the set of all extreme points of $\{a \in M_n \big| a \geq 0, tr(a) = 1\}$ and $PS(M_n)$. In section 3, we find a necessary and sufficient condition for a state on $M_{n_1} \oplus \cdots \oplus M_{n_k}$ to be extended to a pure state on $M_{n_1+\cdots+n_k}$.

2. Relation between states and positive matrices

In this section, we study properties for states and pure states on matrix algebras. For $a = [a_{ij}] \in M_n$, put $tr(a) = \sum_{i=1}^{n} a_{ii}$. In what follows, A denotes the set $\{a \in M_n \mid a \geq 0,\ tr(a) = 1\}$.

THEOREM 2.1. *The following are equivalent:*
(1) $p \in M_n$ *is a projection with rank* 1.
(2) $p \in A$ *is an extreme point of* A.

Proof. (1)⇒(2); Let p be a projection with rank 1. Then there is a vector $v \in \mathbb{C}^n$ such that $pv = v$ and $\|v\| = 1$. Note that $\|a\| \leq tr(a)$ for $a \in A$. Suppose that $p = \lambda a + (1-\lambda)b$ for some $a, b \in A$ and $0 < \lambda < 1$. Since

$$\begin{aligned} 1 &= < v, v >=< pv, v >=< (\lambda a + (1-\lambda)b)v, v > \\ &= \lambda < av, v > +(1-\lambda) < bv, v > \end{aligned}$$

and

$$< av, v >\leq 1, < bv, v >\leq 1,$$

we have

$$< av, v >= 1, < bv, v >= 1.$$

Since $\|av\| \leq 1$ and $\|v\| = 1$, we have $av = bv = v$. Since $pw = \lambda aw + (1-\lambda)bw = 0$ for any $w \in \{v\}^{\perp}$, we have

$$\lambda < aw, w > +(1-\lambda) < bw, w >= 0.$$

Since $a \geq 0$ and $b \geq 0$, we have $< aw, w >\geq 0$ and $< bw, w >\geq 0$. Hence we have $< aw, w >= 0$ and $< bw, w >= 0$. Therefore $a = b = p$ and p is an extreme point of A.

(2)⇒(1); Let $p \in A$ be an extreme point of A. Since p is positive, there are real numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ and orthogonal projections $p_1, p_2, \cdots, p_n$ with rank 1 such that $p = \lambda_1 p_1 +$

$\lambda_2 p_2 + \cdots + \lambda_n p_n$. Since $p_i \in A$ for all i and $1 = tr(p) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, we have $\lambda_1 = 1$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. Therefore p is a projection with rank 1. □

Let E_{ij} denote the matrix with 1 in the (i, j)-entry and 0 elsewhere.

THEOREM 2.2. *Let $f : M_n \to \mathbb{C}$ be a linear functional. Then the following are equivalent:*
(1) f is a state on M_n.
(2) $[f(E_{ij})] \geq 0$ and $\sum_{i=1}^n f(E_{ii}) = 1$.

Proof. (1)⇒(2); For $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n$, let

$$a = \begin{pmatrix} \overline{x_1} \\ \vdots \\ \overline{x_n} \end{pmatrix} (x_1 \ \cdots \ x_n) = \begin{pmatrix} \overline{x_1}x_1 & \cdots & \overline{x_1}x_n \\ \vdots & \cdots & \vdots \\ \overline{x_n}x_1 & \cdots & \overline{x_n}x_n \end{pmatrix}.$$

Since a is positive and f is a state, $f(a) \geq 0$ and

$$f(a) = \sum_{i,j=1}^n f(E_{ij})\overline{x_i}x_j =< [f(E_{ij})]x, x > .$$

Hence $[f(E_{ij})]$ is positive and $1 = f(I_n) = \sum_{i=1}^n f(E_{ii})$.

(2)⇒(1); First, we have $f(I_n) = \sum_{i=1}^n f(E_{ii}) = 1$. Let p be a projection with rank 1. Then there is a vector $x \in \mathbb{C}^n$ with

$$p = \begin{pmatrix} \overline{x_1} \\ \vdots \\ \overline{x_n} \end{pmatrix} (x_1 \ \cdots \ x_n).$$

Hence

$$f(p) = \sum_{ij=1}^n f(E_{ij})\overline{x_i}x_j =< [f(E_{ij})]x, x >\geq 0.$$

Note that for a positive matrix $a \in M_n$, there are non-negative real numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ and projections $p_1, p_2, \cdots, p_n$ with rank 1 such that $a = \lambda_1 p_1 + \cdots + \lambda_n p_n$. Therefore for any positive matrix $a \in M_n$

$$f(a) = f\left(\sum_{i=1}^n \lambda_i p_i\right) = \sum_{i=1}^n \lambda_i f(p_i) \geq 0$$

and thus f is a state on M_n. □

If we define a function $\Phi : St(M_n) \to A$ by $\Phi(f) = [f(E_{ij})]$, Φ is well-defined by Theorem 2.2.

THEOREM 2.3. *The function Φ above satisfies the following:*
(1) *Φ is a one-to-one correspondence between $St(M_n)$ and A.*
(2) *For $0 \leq \lambda \leq 1$, and $f, g \in St(M_n)$, we have*

$$\Phi(\lambda f + (1-\lambda)g) = \lambda\Phi(f) + (1-\lambda)\Phi(g).$$

(3) *f is an extreme point of $St(M_n)$ if and only if $[f(E_{ij})]$ is an extreme point of A.*

Proof. (1) If $\Phi(f) = \Phi(g)$ for $f, g \in St(M_n)$, then $f(E_{ij}) = g(E_{ij})$ for $1 \leq i, j \leq n$. Hence $f = g$, i.e., Φ is injective. For $a = [a_{ij}] \in A$, we associate a linear functional f_a on M_n with a as follows:

$$f_a([x_{ij}]) = \sum_{i,j=1}^{n} a_{ij}x_{ij}.$$

Then $f_a \in St(M_n)$ and $\Phi(f_a) = a$. Hence Φ is a one-to-one correspondence.
(2) Since $\Phi(\lambda f + (1-\lambda)g) = \lambda[f(E_{ij})] + (1-\lambda)[g(E_{ij})]$, we have

$$\Phi(\lambda f + (1-\lambda)g) = \lambda\Phi(f) + (1-\lambda)\Phi(g).$$

(3) It follow directly from (1) and (2). □

COROLLARY 2.4. *Let $f : M_n \to \mathbb{C}$ be a linear functional. Then the following are equivalent:*
(1) *f is a pure state.*
(2) *$[f(E_{ij})]$ is a projection with rank 1.*
(3) *There is a unit vector $v \in \mathbb{C}^n$ such that , for $a \in M_n$,*

$$f(a) = < av, v > .$$

Proof. (1) $\Leftrightarrow$ (2): It follows from Theorem 2.1 and Theorem 2.3.
(2) $\Rightarrow$ (3): Let $[f(E_{ij})]$ be a projection with rank 1. Then there is a unit vector $v \in \mathbb{C}^n$ such that $vv^* = [f(E_{ij})]$ and $f(a) = < av, v >$.
(3) $\Rightarrow$ (2): By elementary calculation, $[f(E_{ij})] = vv^*$ and vv^* is a projection with rank 1. □

3. The Extension of States on $M_n \oplus M_m$

For linear functionals $f : M_n \to \mathbb{C}$ and $g : M_m \to \mathbb{C}$, define

$$f \oplus g : M_n \oplus M_m \to \mathbb{C}$$

by

$$(f \oplus g)(a \oplus b) = f(a) + g(b).$$

Then $f \oplus 0$ and $0 \oplus g$ are obviously states if so are f and g.

Lemma 3.1. *If f and g are pure states, then $f \oplus 0$ and $0 \oplus g$ are pure states.*

Proof. Let $0 < \lambda < 1$ and $f \oplus 0 = \lambda\phi + (1 - \lambda)\psi$ for some states ϕ, ψ on $M_n \oplus M_m$. Define $\phi_1, \psi_1 : M_n \longrightarrow \mathbb{C}$ by $\phi_1(a) = \phi(a \oplus 0)$, $\psi_1(a) = \psi(a \oplus 0)$ and define $\phi_2, \psi_2 : M_m \longrightarrow \mathbb{C}$ by $\phi_2(b) = \phi(0 \oplus b)$, $\psi_2(b) = \psi(0 \oplus b)$. Then $\phi_1, \phi_2, \psi_1, \psi_2$ are positive and

$$f = \lambda\phi_1 + (1 - \lambda)\psi_1, \; 0 = \lambda\phi_2 + (1 - \lambda)\psi_2.$$

Hence $\phi_1 = \psi_1 = f$ and $\phi_2 = \psi_2 = 0$. Thus $f \oplus 0$ is a pure state. Similarly, $0 \oplus g$ is a pure state. □

Let $PS(M_n)$ be the set of all pure states on M_n, and $PS(M_n \oplus M_m)$ be the set of pure states on $M_n \oplus M_m$.

Theorem 3.2.

$$(PS(M_n) \oplus 0) \cup (0 \oplus PS(M_m)) = PS(M_n \oplus M_m).$$

Proof. By Lemma 3.1, $(PS(M_n)\oplus 0)\cup(0\oplus PS(M_m)) \subset PS(M_n\oplus M_m)$. For a pure state f on $M_n \oplus M_m$, define $f_1 : M_n \to \mathbb{C}$ by $f_1(a) = f(a \oplus 0)$ and define $f_2 : M_m \to \mathbb{C}$ by $f_2(b) = f(0\oplus b)$. Then $f = (f_1\oplus 0)+(0\oplus f_2)$. If $f_1(I_n) \neq 0 \neq f_2(I_m)$, then $1 = f(I_n \oplus I_m) = f_1(I_n) + f_2(I_m)$ and

$$f = f_1(I_n)\left(\frac{1}{f_1(I_n)}(f_1 \oplus 0)\right) + f_2(I_m)\left(\frac{1}{f_2(I_m)}(0 \oplus f_2)\right).$$

Since f is a pure state on $M_n \oplus M_m$, $f_1 \equiv 0$ or $f_2 \equiv 0$. If $f_2 \equiv 0$, then f_1 is a pure state. If $f_1 \equiv 0$, then f_2 is a pure state. Therefore

$$PS(M_n \oplus M_m) \subset (PS(M_n) \oplus 0) \cup (0 \oplus PS(M_m)).$$

□

For linear functionals $f : M_n \to \mathbb{C}$ and $g : M_m \to \mathbb{C}$, define $f \otimes g : M_n \otimes M_m \to \mathbb{C}$ by

$$(f \otimes g)(a \otimes b) = f(a)g(b).$$

Then $f \otimes g$ is obviously a state on $M_n \otimes M_m$ if so are f and g .

THEOREM 3.3. *Let $f : M_n \to \mathbb{C}$ and $g : M_m \to \mathbb{C}$ be states. Then the following are equivalent:*
(1) *f and g are pure states.*
(2) *$f \otimes g$ is a pure state.*

Proof. (1)⇒(2): Let f and g be pure states. By Corollary 2.4, there are unit vectors $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ such that $f(a) =< ax, x >$ and $g(b) =< by, y >$. Hence for $a \in M_n$, $b \in M_m$,

$$(f \otimes g)(a \otimes b) =< ax, x >< by, y >=< (a \otimes b)(x \otimes y), x \otimes y > .$$

Thus $f \otimes g$ is a pure state by Corollary 2.4 .
(2)⇒(1): Note that $(f_1 + f_2) \otimes g = f_1 \otimes g + f_2 \otimes g$. Hence if f is not a pure state, then $f \otimes g$ is not a pure state. Similarly, if g is not a pure state, then $f \otimes g$ is not a pure state. □

Let (a, b), $(c, d) \in \mathbb{C}^2$. Then $(a, b) \otimes (c, d) = (ac, ad, bc, bd) \in \mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$. For $v = (\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$ with $\alpha\delta - \beta\gamma \neq 0$, define a pure state h on $M_2 \otimes M_2 = M_4$ by $h(a) =< av, v >$. Since $v \neq x \otimes y$ for any $x, y \in \mathbb{C}^2$, $h \neq f \otimes g$ for any pure states $f, g : M_2 \to \mathbb{C}$. In general, if $m \neq 1$ and $n \neq 1$, then $PS(M_n) \otimes PS(M_m) \subsetneq PS(M_n \otimes M_m)$.

THEOREM 3.4. *For a state $f : M_{n_1} \oplus \cdots \oplus M_{n_k} \to \mathbb{C}$, define*

$$f_1 : M_{n_1} \to \mathbb{C}, \cdots , f_k : M_{n_k} \to \mathbb{C}$$

by

$$f_1(a) = f(a \oplus 0 \oplus \cdots \oplus 0), \cdots , f_k(a) = f(0 \oplus 0 \oplus \cdots \oplus 0 \oplus a).$$

Then the following are equivalent:
(1) *For each i, $rank[f_i(E_{st})] \leq 1$.*
(2) *There is a pure state $g : M_{n_1+\cdots+n_k} \to \mathbb{C}$ such that*

$$g(a_1 \oplus \cdots \oplus a_k) = f(a_1 \oplus \cdots \oplus a_k).$$

Proof. (1)⇒(2); Since f is positive, f_i is positive and so $[f_i(E_{st})]$ is positive. Moreover, since $rank([f_i(E_{st})]) \leq 1$, there exists a vector

$v_i \in \mathbb{C}^{n_i}$ such that $v_i v_i^* = [f_i(E_{st})]$ and $f_i(a) = < av_i, v_i >$.

Put $v = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} \in \mathbb{C}^{n_1+\cdots+n_k}$. Then we have

$$\begin{aligned} f(a_1 \oplus \cdots \oplus a_k) &= f_1(a_1) + \cdots + f_k(a_k) \\ &= < a_1 v_1, v_1 > + \cdots + < a_k v_k, v_k > \\ &= < (a_1 \oplus 0 \oplus \cdots \oplus 0)v, v > + \cdots \\ &\quad + < (0 \oplus \cdots \oplus 0 \oplus a_k)v, v > \\ &= < (a_1 \oplus 0 \oplus \cdots \oplus 0 + \cdots + 0 \oplus \cdots \oplus 0 \oplus a_k)v, v > \\ &= < (a_1 \oplus a_2 \oplus \cdots \oplus a_k)v, v > . \end{aligned}$$

Define $g : M_{n_1+n_2+\cdots+n_k} \to \mathbb{C}$ by $g(a) =< av, v >$. Then $g(a_1 \oplus \cdots \oplus a_k) = f(a_1 \oplus \cdots \oplus a_k)$ and g is a pure state by Corollary 2.4.

(1)⇒(2); By Corollary 2.4, there is a unit vector $v \in \mathbb{C}^{n_1+\cdots+n_k}$ such that $g(a) =< av, v >$ for all $a \in M_{n_1+n_2+\cdots+n_k}$. Put

$$v_1 = (I_{n_1} \oplus 0 \oplus \cdots \oplus 0)v, \cdots , v_k = (0 \oplus 0 \oplus 0 \oplus \cdots \oplus I_{n_k})v.$$

Then $f_i(a) =< av_i, v_i >$ for $a \in M_{n_i}$ and $[f_i(E_{st})] = v_i v_i^*$. Hence $rank[f_i(E_{st})] \leq 1$. □

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Department of Mathematics
University of Seoul
Seoul 130-743, Korea
E-mail: tklee@uoscc.uos.ac.kr