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NEWTON'S METHOD FOR EQUATIONS RELATED TO EXPONENTIAL FUNCTION

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Abstract. For some equation related with exponential function, we seek roots and find the properties of the roots. By using the relation of the roots and attractors, we find a region in the basin of attraction of the attractor at infinity for Newton's method for solving given equation.

1. Introduction

We often need to find the roots of an equation $f(z) = 0$. To be sure, if f is a linear or quadratic polynomial, formulas for writing exact solutions exist and are well known. For general f, Newton's method is a clever numerical procedure for solving equations $f(z) = 0$. For Newton's method we begin with a guess z_0, for a root to the equation $f(z) = 0$. We use the *Newton iteration function*, given by $N_f(z) = z - f(z)/f'(z)$ to find a better approximation $z_1 = N_f(z_0)$. *Newton's method* is to iterate the function N_f, by successively computing $z_2 = N_f(z_1) = N_f(N_f(z_0)), z_3 = N_f(z_2) = N_f(N_f(N_f(z_0)))$ and so on. It is not always true that Newton's method yields approximations that converge to the root (see [7; p. 497]).

The simple roots of $f(z) = 0$ are fixed points of $N_f(z)$ satisfying $N_f(z) = z$ and we call them *attracting fixed points* for $N_f(z)$ or simply *attractors* for $N_f(z)$ since the derivative $N_f'(z) = \frac{f(z)\,f''(z)}{(f'(z)^2)} = 0$ there (see [1; p.214]). Hence, a Newton sequence $\{z_k\}$ given by Newton's method converges to a root of $f(z) = 0$ if z_0 is a proper initial guess. For a simple root z_* satisfying $f(z_*) = 0$, the *basin of attraction* of z_* is the set of all points whose orbits tend to z_*.

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The equation $z^3 - 1 = 0$ has three roots and each of the roots becomes an attractor for Newton's method. The graphical image of dynamics of Newton's method for it is fractal in [5; p. 334] and one of the interesting features of the figure is the boundaries between the three regions and every boundary point is actually next to all three regions. We can conjecture similar behavior for the equation which has infinitely many roots. For example, the equation $e^z - 1 = 0$ has infinitely many roots $z = 2k\pi i$, where k is an integer. These roots become attractors of Newton's method for $e^z - 1 = 0$.

Let

$$M_{\zeta,\alpha}(z) = \exp\left(-\alpha\frac{\zeta + z}{\zeta - z}\right)$$

where $\alpha > 0$ and $|\zeta| = 1$. The equation $M_{\zeta,\alpha}(z) - 1 = 0$ has all its infinite number of roots on the unit circle (see [4]). More properties including Denjoy-Wolff points of $M_{\zeta,\alpha}$ can be found in [2, 3].

In this note, a family of functions

$$E_{A,w_0}(z) = \exp\left(i\,\frac{z\bar{w}_0 - Aw_0}{z - A}\right)$$

where $|A| = 1, w_0 \in \mathbb{C}$ with $\mathrm{Im}w_0 > 0$ is considered. The mapping property of the function E_{A,w_0} can be found in [6; p. 60] and it has singularity at $z = A$. E_{A,w_0} is a generalization of the form of $M_{\zeta,\alpha}$.

In section 2, we check the behavior of the roots by comparing the distances of the adjacent roots and by comparing the distances between the roots and A. By using the relation of the roots and attractors, we find a region in the basin of attraction of the attractor at ∞ for Newton's method for solving given equation.

This note has much importance in the sense that we characterize the dynamical behavior of Newton's method for solving $E_{A,w_0}(z) - 1 = 0$ with given condition and provide the basis for the graphical image.

2. Newton's method for $E_{A,w_0}(z) - 1 = 0$

The roots of the equation $E_{A,w_0}(z) - 1 = 0$ are the attractors of $N_{E_{A,w_0}-1}(z)$. First, we will check the location of them. If $w_0 = \alpha i$ where $\alpha > 0$, then E_{A,w_0} equals $M_{A,\alpha}$ and then we locate the roots of E_{A,w_0} as in [4]. Through this paper we assume that $|A| = 1$ and $w_0 \in \mathbb{C}$ with $\mathrm{Im}w_0 > 0$ without loss of generality.

The equation $E_{A,w_0}(z) - 1 = 0$ has the roots ζ_k where

$$\frac{\zeta_k \bar{w}_0 - A w_0}{\zeta_k - A} = 2k\pi$$

with an integer k. Hence, $\zeta_k = A(2k\pi - w_0)/(2k\pi - \bar{w}_0)$. Since $|\zeta_k| = 1$, the roots ζ_k are located on the unit circle. Therefore, the equation $E_{A,w_0}(z) - 1 = 0$ has infinitely many roots ζ_k and all of them lie on the unit circle.

Theorem 1. *As $k \to \infty$, the roots ζ_k of the equation $E_{A,w_0}(z) - 1 = 0$ approach to A in a counterclockwise direction and the roots ζ_{-k} approach to A in a clockwise direction. For any positive integer k, the followings hold:*

i) the roots ζ_k and ζ_{-k} are symmetric about the ray of angle ϕ iff $Rew_0 = 0$

ii) the root ζ_{-k} is closer to the singular point A than the root ζ_k is to A iff $Rew_0 > 0$.

Proof. As $k \to \infty$, $|\zeta_k - A| = 2\mathrm{Im}w_0/|2k\pi - \bar{w}_0|$ approaches to 0. Since $\zeta_k/A = \frac{2k\pi - w_0}{2k\pi - \bar{w}_0}$, we see that the roots ζ_k approach to A in a counterclockwise direction and the roots ζ_{-k} approach to A in a clockwise direction as $k \to \infty$ by considering the $Arg(\zeta_k/A)$.

Since $|\zeta_k - A| = 2\mathrm{Im}w_0/|2k\pi - \bar{w}_0|$, we see that

$$\begin{aligned} |\zeta_k - A| > |\zeta_{-k} - A| \quad & \text{if} \quad |2k\pi - \bar{w}_0| < |-2k\pi - \bar{w}_0| \\ & \text{if} \quad k(\mathrm{Re}w_0) > 0. \end{aligned}$$

Hence the inequality $|\zeta_k - A| > |\zeta_{-k} - A|$ holds for any positive integer k iff $Rew_0 > 0$ and the equality $|\zeta_k - A| = |\zeta_{-k} - A|$ holds for any integer k iff $Rew_0 = 0$.

It means that for any positive integer k, the roots ζ_k and ζ_{-k} are symmetric about the ray of angle ϕ where $A = e^{i\phi}$ iff $\mathrm{Re}w_0 = 0$ and the root ζ_{-k} is closer to the singular point A than the root ζ_k is to A iff $\mathrm{Re}w_0 > 0$. $\square$

Next, we study the behavior of the roots of the equation $E_{A,w_0}(z) - 1 = 0$ more carefully. To do this we need the following two lemmas.

Lemma 2. *Let* ζ_k *be the roots of the equation* $E_{A,w_0}(z) - 1 = 0$. *If* $2(k-1)\pi < Rew_0 < 2k\pi$, *then* $|\zeta_k - \zeta_{k-1}|$ *is the biggest among the distance between the adjacent roots. If* $Rew_0 = 2k\pi$, *then* $|\zeta_k - \zeta_{k-1}| = |\zeta_{k+1} - \zeta_k|$ *is the biggest among the distance between the adjacent roots.*

Proof. Since

$$|\zeta_k - \zeta_{k-1}| = |\frac{2k\pi - w_0}{2k\pi - \bar{w}_0} - \frac{2(k-1)\pi - w_0}{2(k-1)\pi - \bar{w}_0}|,$$

we can compare the distances between two adjacent roots by calculating

$$\begin{aligned} &|\zeta_k - \zeta_{k-1}|^2 - |\zeta_{k+1} - \zeta_k|^2 \\ &= (4\pi \mathrm{Im} w_0)^2 \frac{8\pi(2k\pi - \mathrm{Re} w_0)}{|(2(k-1)\pi - \bar{w}_0)(2k\pi - \bar{w}_0)(2(k+1)\pi - \bar{w}_0)|^2}. \end{aligned}$$

Hence, $|\zeta_k - \zeta_{k-1}| > |\zeta_{k+1} - \zeta_k|$ iff $2k\pi > \mathrm{Re}w_0$.

Therefore, if $2(k-1)\pi < \mathrm{Re}w_0 \le 2k\pi$, then

$$\begin{aligned} &\cdots < |\zeta_{k-2} - \zeta_{k-3}| < |\zeta_{k-1} - \zeta_{k-2}| < |\zeta_k - \zeta_{k-1}| \\ \text{and} \quad &\cdots < |\zeta_{k+2} - \zeta_{k+1}| < |\zeta_{k+1} - \zeta_k| \le |\zeta_k - \zeta_{k-1}| \end{aligned}$$

hold and it induces the desired conclusion . □

Lemma 3. *Let* ζ_k *be the roots of the equation* $E_{A,w_0}(z) - 1 = 0$. *If* $(2k-1)\pi < Rew_0 < (2k+1)\pi$, *then* ζ_k *is the farthest root from* A. *If* $Rew_0 = (2k-1)\pi$, *then* ζ_k *and* ζ_{k-1} *are the farthest roots from* A.

Proof. Since

$$|\zeta_k - A| = \frac{2\mathrm{Im}w_0}{|2k\pi - \bar{w}_0|},$$

we can compare the distances between the roots and A by calculating

$$|\zeta_k - A|^2 - |\zeta_{k-1} - A|^2 = (2\mathrm{Im}w_0)^2 \left(\frac{-4\pi((2k-1)\pi - \mathrm{Re}w_0)}{|2k\pi - \bar{w}_0|^2 \, |2(k-1)\pi - \bar{w}_0|^2} \right).$$

Hence, $|\zeta_k - A| > |\zeta_{k-1} - A|$ iff $(2k-1)\pi < \mathrm{Re}w_0$.

It means that if $(2k-1)\pi \le \mathrm{Re}w_0 < (2k+1)\pi$, then

$$\begin{aligned} &|\zeta_k - A| \ge |\zeta_{k-1} - A| > |\zeta_{k-2} - A| > \cdots \\ \text{and} \quad &|\zeta_k - A| > |\zeta_{k+1} - A| > |\zeta_{k+2} - A| > \cdots \end{aligned}$$

hold and it leads to the conclusion. □

By combining Lemma 2 and Lemma 3, we conclude

THEOREM 4. *Let ζ_k be the roots of the equation $E_{A,w_0}(z) - 1 = 0$.*

If $2(k-1)\pi < Rew_0 < (2k-1)\pi$, then $|\zeta_k - \zeta_{k-1}|$ is the biggest among the distance between the adjacent roots and ζ_{k-1} is the farthest root from A.

If $(2k-1)\pi < Rew_0 < 2k\pi$, then $|\zeta_k - \zeta_{k-1}|$ is the biggest among the distance between the adjacent roots and ζ_k is the farthest root from A.

If $Rew_0 = (2k-1)\pi$, then $|\zeta_k - \zeta_{k-1}|$ is the biggest among the distance between the adjacent roots and ζ_k and ζ_{k-1} are the farthest roots from A.

If $Rew_0 = 2k\pi$, then $|\zeta_k - \zeta_{k-1}| = |\zeta_{k+1} - \zeta_k|$ is the biggest among the distance between the adjacent roots and ζ_k is the farthest root from A.

To understand the location of the roots, we consider the following two examples.

EXAMPLE 5. If $w_0 = i$ and $A = 1$, then $\zeta_0 = -1$ and $\zeta_k = \frac{2k\pi - i}{2k\pi + i} = \frac{(2k\pi)^2 - 1 - 4k\pi i}{(2k\pi)^2 + 1}$. Hence $-\pi/2 < \text{Arg}\,\zeta_k < 0$ and $0 < \text{Arg}\,\zeta_{-k} < \pi/2$ for any positive integer k. Also, $\text{Arg}\,\zeta_k \to 0^-$ and $\text{Arg}\,\zeta_{-k} \to 0^+$ as $k \to \infty$. Note that $\zeta_{-k} = \bar{\zeta}_k$ in this case. By Theorem 4, $|\zeta_0 - \zeta_{-1}| = |\zeta_1 - \zeta_0|$ is the biggest among the distance between the adjacent roots and ζ_0 is the farthest root from A. $\square$

EXAMPLE 6. If $w_0 = 5(1+i)$ and $A = 1$, then $\zeta_0 = i$ and $\zeta_k = \frac{(2k\pi - 5)^2 - 5^2 - 10(2k\pi - 5)i}{(2k\pi - 5)^2 + 5^2}$. Hence for $k \geq 2, -\pi/2 < \text{Arg}\,\zeta_k < 0$ and for any $k \geq 1, 0 < \text{Arg}\,\zeta_{-k} < \pi/2$. Also $\text{Arg}\,\zeta_k \to 0^-$ and $\text{Arg}\,\zeta_{-k} \to 0^+$ as $k \to \infty$. By Theorem 1, for any positive integer k $|\zeta_{-k} - 1| < |\zeta_k - 1|$.

By Theorem 4, $|\zeta_1 - \zeta_0|$ is the biggest among the distance between the adjacent roots and ζ_1 is the farthest root from A. $\square$

In Theorem 7 we check the behavior of the Newton sequence near ∞.

THEOREM 7. *The region satisfying*

$$|z| > 2 + \frac{4Imw_0}{1 - e^{-\text{Im}w_0/3}} \tag{1}$$

is in the basin of attraction of the attractor at infinity for Newton's method for the equation $E_{A,w_0}(z) - 1 = 0$.

Proof. Newton iteration function is

$$
\begin{aligned}
N_{E_{A,w_0}-1}(z) &= z - \frac{E_{A,w_0}(z)-1}{E'_{A,w_0}(z)} \\
&= z + \frac{(z-A)^2}{2A\mathrm{Im}w_0}\left[1-\exp\left(-i\,\frac{z\bar{w}_0 - Aw_0}{z-A}\right)\right].
\end{aligned}
$$

By letting $F(z) = 1/N_{E_{A,w_0}-1}(1/z)$, we check that $F(z) \to 0$ and $F'(z) \to 0$ as $z \to 0$. Hence $z = 0$ is an attracting fixed point for $F(z)$ and so $z = \infty$ is an attracting fixed point for $N_{E_{A,w_0}-1}(z)$.

Now, we prove the region satisfying (1) is in the basin of attraction of the attractor at infinity for $N_{E_{A,w_0}-1}(z)$.

If

$$
|z| > 2 + \frac{4\mathrm{Im}w_0}{1-e^{-\mathrm{Im}w_0/3}},
$$

then $|z| > 2$. Since

$$
\begin{aligned}
\mathrm{Re}\left[-i\,\frac{z\bar{w}_0 - Aw_0}{z-A}\right] &= \mathrm{Im}\left[\frac{(z\bar{w}_0 - Aw_0)(\bar{z}-\bar{A})}{|z-A|^2}\right] \\
&= -\frac{\mathrm{Im}w_0(|z|^2-1)}{|z-A|^2},
\end{aligned}
$$

it holds

$$
\begin{aligned}
\left|\exp\left(-i\,\frac{z\bar{w}_0 - Aw_0}{z-A}\right)\right| &= \exp\left(\mathrm{Re}\left[-i\,\frac{z\bar{w}_0 - Aw_0}{z-A}\right]\right) \\
&\le \exp\left(-\frac{\mathrm{Im}w_0(|z|^2-1)}{(|z|+1)^2}\right) \\
&< \exp\left(-\frac{\mathrm{Im}w_0}{3}\right)
\end{aligned}
$$

when $|z| > 2$. Note that we assumed $\mathrm{Im}w_0 > 0$ through this paper.

Therefore if (1) holds, then

$$
\begin{aligned}
|N_{E_{A,w_0}-1}(z)| &\geq \frac{|z-A|^2}{2\,\mathrm{Im}w_0}\left|1-\exp\left(-i\,\frac{z\bar{w}_0-Aw_0}{z-A}\right)\right| - |z| \\
&> \frac{(|z|-1)^2}{2\,\mathrm{Im}w_0}(1-e^{-\mathrm{Im}w_0/3}) - |z| \\
&> |z|\left(\frac{1-e^{-\mathrm{Im}w_0/3}}{2\,\mathrm{Im}w_0}|z| - \frac{1-e^{-\mathrm{Im}w_0/3}+\mathrm{Im}w_0}{\mathrm{Im}w_0}\right) \\
&> |z|.
\end{aligned}
$$

It implies that Newton sequence generated by an initial value z_0 in the region satisfying (1) diverges to ∞. So the region satisfying (1) is contained in the basin of attraction of the attractor at ∞. □

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