

ENTROPY NUMBERS OF (R, W) -NUCLEAR OPERATORS ACTING BETWEEN BANACH SPACES OF CERTAIN WEAK TYPE

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ABSTRACT. We characterize (r, w) -nuclear operators acting from a Banach space whose dual has weak type q into a Banach space of weak type p by the asymptotic behaviour of their entropy numbers.

1. Introduction

The theory of the so-called entropy numbers was introduced by A. Pietsch [12]. Afterwards a lot of results concerning the behaviour of entropy numbers of certain classes of operators were established (cf. [4], [6], [10]). We only remind of diagonal operators acting between Lorentz sequence spaces as well as embedding maps between Besov function spaces.

B. Carl [4] characterized diagonal operators acting between Lorentz sequence spaces by their entropy numbers.

In [5] B. Carl considered operators $S : \ell_q \rightarrow X$ admitting a factorization through ℓ_1 , $S : \ell_q \xrightarrow{D} \ell_1 \xrightarrow{B} X$, where D is a diagonal operator generated by a sequence belonging to a Lorentz sequence space and B is an arbitrary bounded operator. He characterized these operators in terms of their entropy numbers under the hypothesis that X is a Banach space of type p . By using this result, he [6] showed that the sequence of entropy numbers of r -nuclear operators acting from a Banach space L_p into a Banach space L_q belongs to the Lorentz sequence space.

In [10] T. Kühn dealt with operators $T : X \rightarrow \ell_q$ factorizing through ℓ_∞ , $T : X \xrightarrow{B} \ell_\infty \xrightarrow{D} \ell_q$, where B is an arbitrary bounded operator

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and D is a diagonal operator generated by a sequence belonging to a Lorentz sequence space. He estimated the asymptotic behaviour of the entropy numbers of these operators under the assumption that the dual of a Banach space X has type p . And then he applied this result to r -nuclear operators acting from a Banach space whose dual has type q into a Banach space of type p .

In this paper we determine the “degree of compactness” of (r, w) -nuclear operators acting from a Banach space whose dual has weak type q into a Banach space of weak type p by means of entropy numbers. Here, we present Defant and Junge’s approach [7].

2. Definitions and Notation

We present some of the definitions and notation to be used. Throughout this paper X and Y denote Banach spaces.

Let (A_0, A_1) be a couple of quasi-Banach spaces. We consider the functional $K(t, a, A_0, A_1) = K(t, a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a_0 \in A_0, a_1 \in A_1, a = a_0 + a_1\}$ on $A_0 + A_1$. If $0 < \theta < 1$ and $0 < q \leq \infty$ then the real interpolation space $(A_0, A_1)_{\theta, q}$ consists of all elements $a \in A_0 + A_1$ which have a finite quasi-norm

$$\|a\|_{(A_0, A_1)_{\theta, q}} = \|a\|_{\theta, q} = \begin{cases} \left(\int_0^\infty [t^{-\theta} K(t, a)]^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \sup_t [t^{-\theta} K(t, a)] & \text{if } q = \infty. \end{cases}$$

Notation. (1) The dual of a Banach space X is denoted by X^* .

(2) For $1 < p < \infty$, the conjugate of p is denoted by p' , i.e., $1/p + 1/p' = 1$.

(3) The closed unit ball of a Banach space X is denoted by B_X .

(4) $\mathcal{B}(X, Y)$ denotes the set of all bounded linear operators from X into Y .

(5) $\mathcal{F}(X, Y)$ denotes the set of all finite rank operators from X into Y .

(6) The dual operator of an operator T is denoted by T^* .

(7) $\text{Vol}(\cdot)$ denotes the Lebesgue measure on \mathbb{R}^n .

For every operator $T \in \mathcal{B}(X, Y)$ the n -th outer entropy number $e_n(T)$ is defined to be the infimum of all $\epsilon \geq 0$ such that there are elements $y_1, \dots, y_q \in Y$ with $q \leq 2^{n-1}$ and $T(B_X) \subseteq \bigcup_{i=1}^q \{y_i + \epsilon B_Y\}$.

The n -th approximation number of $T \in \mathcal{B}(X, Y)$ is defined by

$$a_n(T) = \inf\{\|T - L\| : L \in \mathcal{F}(X, Y), \text{rank}(L) < n\}.$$

The n -th Gel'fand number of $T \in \mathcal{B}(X, Y)$ is defined by

$$c_n(T) = \inf\{\|T J_M^X\| : \text{codim}(M) < n\},$$

where J_M^X denotes the canonical injection from the subspace M into X .

The n -th Kolmogorov number of $T \in \mathcal{B}(X, Y)$ is defined by

$$d_n(T) = \inf\{\|Q_N^Y T\| : \dim(N) < \infty\},$$

where Q_N^Y denotes the canonical surjection from Y onto the quotient space Y/N .

The n -th Weyl number of $T \in \mathcal{B}(X, Y)$ is defined by

$$x_n(T) = \sup\{a_n(TU) : U \in \mathcal{B}(\ell_2, X), \|U\| \leq 1\}.$$

The n -th Grothendieck number of $T \in \mathcal{B}(X, Y)$ is defined by

$$\Gamma_n(T) = \sup\{|\det(\langle Tx_i, y_j^* \rangle)|^{1/n} : (x_k)_{k=1}^n \subset B_X, (y_k^*)_{k=1}^n \subset B_{Y^*}\}.$$

The n -th volume number of $T \in \mathcal{B}(X, Y)$ is defined by

$$v_n(T) = \sup\left\{\left(\frac{\text{Vol}(T(B_M))}{\text{Vol}(B_N)}\right)^{1/n} : M \subset X, T(M) \subset N \subset Y, \dim M = \dim N = n\right\}.$$

The n -th volume ratio number of $T \in \mathcal{B}(X, Y)$ is defined by

$$vr_n(T) = \sup\left\{\left(\frac{\text{Vol}(Q_N^Y(T(B_X)))}{\text{Vol}(B_{Y/N})}\right)^{1/n} : N \subset Y, \text{codim } N = n\right\}.$$

For an operator $U \in \mathcal{B}(\ell_2^n, X)$, we define

$$\ell(U) = \left(\int_{\mathbb{R}^n} \|Ux\|^2 d\gamma_n(x)\right)^{1/2},$$

where γ_n is the canonical Gaussian probability measure on \mathbb{R}^n .

For an operator $V \in \mathcal{B}(X, \ell_2^n)$, we set

$$\ell^*(V) = \sup\{|\text{tr}(VU)| : U \in \mathcal{B}(\ell_2^n, X), \ell(U) \leq 1\}.$$

A Banach space X is called a weak type p space if there is a constant C such that for all n and all operators $V \in \mathcal{B}(X, \ell_2^n)$, we have $\sup_k k^{1/p'} a_k(V) \leq C \cdot \ell^*(V)$. The smallest constant C for which this holds will be denoted by $wT_p(X)$.

A Banach space X is called K -convex if there is a constant C such that for every n and every operator $V \in \ell^*(X, \ell_2^n)$, we have $\ell(V^*) \leq C \cdot \ell^*(V)$. In this case, we define the K -convexity constant as $K(X) = \inf C$, where the infimum is taken over all constants C satisfying the above inequality.

If $x = (\xi_i)$ is a bounded sequence then we put $s_n(x) = \inf\{\sigma \geq 0 : \text{card}\{i : |\xi_i| \geq \sigma\} < n\}$. $(s_n(x))$ is called the non-increasing rearrangement of x . Let $0 < r < \infty$ and $0 < w \leq \infty$. Then the Lorentz sequence

space $\ell_{r,w}$ consists of all sequences $x = (\xi_i)$ having a finite quasi-norm

$$\|x\|_{r,w} = \begin{cases} (\sum_{n=1}^{\infty} [n^{1/r-1/w} s_n(x)]^w)^{1/w} & \text{if } 0 < w < \infty, \\ \sup_n [n^{1/r} s_n(x)] & \text{if } w = \infty. \end{cases}$$

For $s \in \{e, a, c, d, x, v, vr\}$, an operator $T \in \mathcal{B}(X, Y)$ is said to be of s -type $\ell_{r,w}$ if $(s_n(T)) \in \ell_{r,w}$. The set of these operators is denoted by $\mathcal{L}_{r,w}^{(s)}(X, Y)$. For $T \in \mathcal{L}_{r,w}^{(s)}(X, Y)$, we define $\|T\|_{\mathcal{L}_{r,w}^{(s)}} = \|(s_n(T))\|_{r,w}$ [13].

Let $0 < r < 1$ and $0 < w \leq \infty$. An operator $T \in \mathcal{B}(X, Y)$ is said to be (r, w) -nuclear if it can be written in the form $T = \sum_{i=1}^{\infty} \tau_i x_i^* \otimes y_i$ with (x_i^*) in B_{X^*} , (y_i) in B_Y and $(\tau_i) \in \ell_{r,w}$. The set of these operators is denoted by $T \in \mathcal{N}_{r,w}(X, Y)$. For $T \in \mathcal{N}_{r,w}(X, Y)$, we define a quasi-norm

$$\nu_{r,w}(T) = \begin{cases} \inf(\sum_{n=1}^{\infty} [n^{1/r-1/w} \tau_n]^w)^{1/w} & \text{if } 0 < w < \infty, \\ \inf(\sup_n [n^{1/r} \tau_n]) & \text{if } w = \infty, \end{cases}$$

where the infimum is taken over all (r, w) -nuclear representations such that $\tau_1 \geq \tau_2 \geq \dots \geq 0$.

3. Results

By using Carl's proof [5] and the generalized Carl-Maurey inequality, we estimate the entropy quasi-norm of operators acting from ℓ_1^m into a Banach space of weak type p .

LEMMA 1. *Let X be of weak type p , $1 \leq p < 2$, and $S \in \mathcal{B}(\ell_1^m, X)$. Then there exists a constant $C > 0$ such that $\sup_{1 \leq k < \infty} k^{1/s} e_k(S) \leq C \|S\| m^{1/s-1/p'}$ for $s < p'$ and $m = 1, 2, \dots$*

Proof. We invoke Carl-Maurey inequality [8] to infer that there exists a constant $C \geq 0$ such that

$$\begin{aligned} \sup_{1 \leq k \leq m} k^{1/s} e_k(S) &\leq C \|S\| \sup_{1 \leq k \leq m} k^{1/s-1/p'} [1 + \ln(\frac{m}{k})]^{1/p'} \\ &\leq C_0 \|S\| m^{1/s-1/p'} \quad \text{for } s < p'. \end{aligned}$$

Now we estimate $\sup_{k>m} k^{1/s} e_k(S)$. Let I_m denote the identity operator on ℓ_1^m . Using the multiplicativity of the entropy numbers we obtain that

$$\begin{aligned} \sup_{k>m} k^{1/s} e_k(S) &= \sup_{k \geq 1} (m+k)^{1/s} e_{m+k}(S) \\ &\leq \sup_{k \geq 1} (m+k)^{1/s} e_m(S) e_k(I_m) \leq e_m(S) \sup_{k \geq 1} 2^{1/s} (m^{1/s} + k^{1/s}) e_k(I_m) \\ &\leq 2^{1/s} m^{1/s} e_m(S) + 2^{1/s} e_m(S) \sup_{k \geq 1} k^{1/s} e_k(I_m). \end{aligned}$$

Applying proposition 12.1.13 of [12] we derive that

$$\begin{aligned} \sup_{k \geq 1} k^{1/s} e_k(I_m) &\leq \left(\sum_{k=1}^{\infty} e_k^s(I_m) \right)^{1/s} \leq 4 \left(\sum_{k=1}^{\infty} (2^{-(k-1)/2m})^s \right)^{1/s} \\ &\leq 4 \frac{1}{(1 - 2^{-s/2m})^{1/s}} \leq 4 \frac{2^{1/2m}}{(2^{s/2m} - 1)^{1/s}} \\ &\leq 4 \frac{2^{1/2m}}{(s/2m)^{1/s} (\ln 2)^{1/s}} \leq 8 \frac{2^{1/s} m^{1/s}}{(s \ln 2)^{1/s}}. \end{aligned}$$

It takes another appeal to Carl-Maurey inequality [8] to yield that

$$\begin{aligned} \sup_{k>m} k^{1/s} e_k(S) &\leq e_m(S) m^{1/s} 2^{1/s} \left[1 + \frac{8 \cdot 2^{1/s}}{(s \ln 2)^{1/s}} \right] \\ &\leq C_1 \|S\| m^{1/s-1/p'} 2^{1/s} \left[1 + \frac{8 \cdot 2^{1/s}}{(s \ln 2)^{1/s}} \right] \leq C_2 \|S\| m^{1/s-1/p'}. \end{aligned}$$

Combining this with the above estimate we see that

$$\begin{aligned} \sup_{1 \leq k < \infty} k^{1/s} e_k(S) &\leq \sup_{1 \leq k \leq m} k^{1/s} e_k(S) + \sup_{k > m} k^{1/s} e_k(S) \\ &\leq C_3 \|S\| m^{1/s-1/p'} \quad \text{for } s < p'. \end{aligned}$$

□

Applying this lemma and Carl's proof [4], [5] we intend to characterize operators of the form SD_σ , where $D_\sigma : \ell_q \rightarrow \ell_1$ is a diagonal operator generated by a sequence belonging to a Lorentz sequence space and $S : \ell_1 \rightarrow X$ is an arbitrary operator with the image in a Banach space of weak type p , in terms of entropy numbers.

PROPOSITION 1. Let X be of weak type p , $1 \leq p < 2$, $S \in \mathcal{B}(\ell_1, X)$ and let $D_\sigma \in \mathcal{B}(\ell_1, \ell_1)$ be a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r,t}$, where $0 < r < \infty$ and $0 < t \leq \infty$. Then $SD_\sigma \in \mathcal{L}_{s,t}^{(e)}(\ell_1, X)$ for $1/s = 1/r + 1/p'$.

Proof. First we shall show that if $\sigma \in \ell_{r,\infty}$ then $SD_\sigma \in \mathcal{L}_{s,\infty}^{(e)}(\ell_1, X)$, where $1/s = 1/r + 1/p'$ (*).

There is no loss in assuming that $|\sigma_1| \geq |\sigma_2| \geq \dots \geq 0$. We define canonical operators $J_k \in \mathcal{B}(\ell_1^{2^k}, \ell_1)$ and $Q_k \in \mathcal{B}(\ell_1, \ell_1^{2^k})$ by

$$\begin{aligned} J_k(\xi_1, \dots, \xi_{2^k}) &= (0, \dots, 0, \xi_1, \dots, \xi_{2^k}, 0, \dots), \\ Q_k(\xi_1, \dots, \xi_k, \dots) &= (\xi_{2^k}, \dots, \xi_{2^{k+1}-1}), \end{aligned}$$

respectively, for $k \geq 0$. Let $M_k \in \mathcal{B}(\ell_1^{2^k}, \ell_1^{2^k})$ be the operator defined by $M_k(\eta_1, \dots, \eta_{2^k}) = (\sigma_{2^k} \eta_1, \dots, \sigma_{2^{k+1}-1} \eta_{2^k})$ for $k \geq 0$. Then $D_\sigma = \sum_{k=0}^{\infty} J_k M_k Q_k$ and so $SD_\sigma = \sum_{k=0}^{\infty} S J_k M_k Q_k$. Taking account of the fact that $\mathcal{L}_{s,\infty}^{(e)}$ admits an equivalent α -norm and using lemma 1 we obtain

$$\begin{aligned} \left\| \sum_{k=0}^{m-1} S J_k M_k Q_k \right\|_{\mathcal{L}_{s,\infty}^{(e)}} &\leq C_0 \left(\sum_{k=0}^{m-1} \|S J_k M_k Q_k\|_{\mathcal{L}_{s,\infty}^{(e)}}^\alpha \right)^{1/\alpha} \\ &\leq C_0 \left(\sum_{k=0}^{m-1} \|S J_k\|_{\mathcal{L}_{s,\infty}^{(e)}}^\alpha \|M_k\|^\alpha \|Q_k\|^\alpha \right)^{1/\alpha} \\ &\leq C_0 \left(\sum_{k=0}^{m-1} \|S J_k\|_{\mathcal{L}_{s,\infty}^{(e)}}^\alpha |\sigma_{2^k}|^\alpha \right)^{1/\alpha} \\ &\leq C_1 \left(\sum_{k=0}^{m-1} \|S J_k\|^\alpha 2^{\alpha k(1/s-1/p')} |\sigma_{2^k}|^\alpha \right)^{1/\alpha} \\ &\leq C_1 \|S\| \left(\sum_{k=0}^{m-1} 2^{\alpha k(1/s-1/p')} |\sigma_{2^k}|^\alpha \right)^{1/\alpha} \quad \text{for } s < p'. \end{aligned}$$

We assume that $\sigma = (\sigma_i) \in \ell_{r,\infty}$. Then we get

$$\begin{aligned} \left\| \sum_{k=0}^{m-1} S J_k M_k Q_k \mid \mathcal{L}_{s,\infty}^{(e)} \right\| &\leq C_2 \|S\| \left(\sum_{k=0}^{m-1} 2^{\alpha k(1/s-1/p')} 2^{-\alpha k/r} \right)^{1/\alpha} \\ &\leq C_3 \|S\| 2^{m(1/s-1/p'-1/r)} \quad \text{for } 1/s > 1/r + 1/p'. \end{aligned}$$

This yields $e_{2^{m-1}}(\sum_{k=0}^{m-1} S J_k M_k Q_k) \leq C_4 \|S\| 2^{-m(1/r+1/p')}$. In order to estimate $\|\sum_{k=m}^{\infty} S J_k M_k Q_k \mid \mathcal{L}_{s,\infty}^{(e)}\|$ we choose s such that $1/p' < 1/s < 1/r + 1/p'$. By arguing similarly as above, we establish the following estimation

$$\begin{aligned} \left\| \sum_{k=m}^{\infty} S J_k M_k Q_k \mid \mathcal{L}_{s,\infty}^{(e)} \right\| &\leq C_5 \|S\| \left(\sum_{k=m}^{\infty} 2^{\alpha k(1/s-1/r-1/p')} \right)^{1/\alpha} \\ &\leq C_5 \|S\| 2^{m(1/s-1/r-1/p')} \cdot \left(\sum_{k=0}^{\infty} 2^{\alpha k(1/s-1/r-1/p')} \right)^{1/\alpha} \\ &\leq C_6 \|S\| 2^{m(1/s-1/r-1/p')}. \end{aligned}$$

This implies $e_{2^{m-1}}(\sum_{k=m}^{\infty} S J_k M_k Q_k) \leq C_7 \|S\| 2^{-m(1/r+1/p')}$. From the additivity of the entropy numbers it follows that

$$\begin{aligned} e_{2^m}(SD_\sigma) &\leq e_{2^{m-1}}\left(\sum_{k=0}^{m-1} S J_k M_k Q_k\right) + e_{2^{m-1}}\left(\sum_{k=m}^{\infty} S J_k M_k Q_k\right) \\ &\leq C_8 \|S\| 2^{-m(1/r+1/p')}. \end{aligned}$$

If n is a natural number we take m so that $2^m \leq n < 2^{m+1}$. An appeal to the monotonicity of the entropy numbers establishes that $e_n(SD_\sigma) \leq e_{2^m}(SD_\sigma) \leq C_9 \|S\| n^{-1/r-1/p'}$. Hence $SD_\sigma \in \mathcal{L}_{s,\infty}^{(e)}(\ell_1, X)$ for $1/s = 1/r + 1/p'$, which verifies (*).

Now we use real interpolation to derive the desired assertion. Given r with $0 < r < \infty$ we can find r_0, r_1 and θ such that $0 < r_0 < r_1 < \infty$, $0 < \theta < 1$ and $1/r = (1-\theta)/r_0 + \theta/r_1$. We consider the operator \mathcal{T} transforming every sequence σ into the composition operator SD_σ . By (*), $\mathcal{T} : \ell_{r_i,\infty} \rightarrow \mathcal{L}_{s_i,\infty}^{(e)}(\ell_1, X)$, where $1/s_i = 1/r_i + 1/p'$, $i = 0, 1$, are both bounded linear operators. Since $1/r = (1-\theta)/r_0 + \theta/r_1$,

theorem 5.3.1. of [1] tells us that $(\ell_{r_0, \infty}, \ell_{r_1, \infty})_{\theta, t} = \ell_{r, t}$. The well-known interpolation formula concerning entropy ideals allows us to obtain $(\mathcal{L}_{s_0, \infty}^{(e)}(\ell_1, X), \mathcal{L}_{s_1, \infty}^{(e)}(\ell_1, X))_{\theta, t} \subseteq \mathcal{L}_{s, t}^{(e)}(\ell_1, X)$, where $1/s = (1 - \theta)/s_0 + \theta/s_1 = 1/r + 1/p'$. We apply the interpolation theorem [1] to deduce that $\mathcal{T} : \ell_{r, t} \rightarrow \mathcal{L}_{s, t}^{(e)}(\ell_1, X)$ is also bounded. This ends the proof of the proposition. \square

PROPOSITION 2. *Let X be of weak type p , $1 \leq p < 2$, $S \in \mathcal{B}(\ell_1, X)$ and let $D_\sigma \in \mathcal{B}(\ell_q, \ell_1)$ be a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r, t}$, where $0 < r < \infty$, $0 < t \leq \infty$, $1 \leq q \leq \infty$ and $1/r + 1/q > 1$. Then $SD_\sigma \in \mathcal{L}_{s, t}^{(e)}(\ell_q, X)$ provided that $1/s = 1/r + 1/q - 1/p$.*

Proof. The assumptions on q and r guarantee that we can choose r_0 and r_1 such that $0 < r_0, r_1 < \infty$, $1/r = 1/r_0 + 1/r_1$ and $1/r_1 + 1/q > 1$. Then we can split $\sigma = \tau \circ \mu$ with $\mu \in \ell_{r_1, \infty}$ and $\tau \in \ell_{r_0, t}$. As a result the operator $D_\sigma \in \mathcal{B}(\ell_q, \ell_1)$ can be factorized with diagonal operators D_μ and D_τ as $D_\sigma : \ell_q \xrightarrow{D_\mu} \ell_1 \xrightarrow{D_\tau} \ell_1$. Since $1/r_1 > \max(1 - 1/q, 0)$, we use a result of Carl [4] to see that $D_\mu \in \mathcal{L}_{s_1, \infty}^{(e)}(\ell_q, \ell_1)$ for $1/s_1 = 1/r_1 + 1/q - 1$. An appeal to proposition 1 reveals that $SD_\tau \in \mathcal{L}_{s_0, t}^{(e)}(\ell_1, X)$ for $1/s_0 = 1/r_0 + 1/p'$. Using the multiplication theorem for the entropy ideals we derive that $SD_\sigma = SD_\tau D_\mu \in \mathcal{L}_{s_0, t}^{(e)} \circ \mathcal{L}_{s_1, \infty}^{(e)}(\ell_q, X) \subseteq \mathcal{L}_{s, t}^{(e)}(\ell_q, X)$ for $1/s = 1/s_0 + 1/s_1 = 1/r + 1/q - 1/p$. \square

In the next proposition we describe the dual situation.

PROPOSITION 3. *Let X^* be of weak type p , $1 \leq p < 2$, $R \in \mathcal{B}(X, \ell_\infty)$ and let $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_q)$ be a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r, t}$, where $1 \leq q \leq \infty$, $0 < r < q \leq \infty$ and $0 < t \leq \infty$. Then $D_\sigma R \in \mathcal{L}_{s, t}^{(e)}(X, \ell_q)$ for $1/s = 1/r + 1/q' - 1/p$.*

Proof. We start with the case $1 < q < \infty$. For $R \in \mathcal{B}(X, \ell_\infty)$, we denote the restriction of R^* to ℓ_1 by S . Since X^* is of weak type p and $1/r + 1/q' > 1$, it follows from proposition 2 that $SD_\sigma \in \mathcal{L}_{s, t}^{(e)}(\ell_{q'}, X^*)$ with $1/s = 1/r + 1/q' - 1/p$. The K -convexity of $\ell_{q'}$ enables us to invoke a result due to Bourgain, Pajor, Szarek and Tomczak-Jaegermann [2] to get that there exists a constant $C \geq 0$ such that

$\|D_\sigma R|_{\mathcal{L}_{s,t}^{(e)}}\| = \|D_\sigma S^*|_X|_{\mathcal{L}_{s,t}^{(e)}}\| \leq \|D_\sigma S^*|_{\mathcal{L}_{s,t}^{(e)}}\| \leq C \|SD_\sigma|_{\mathcal{L}_{s,t}^{(e)}}\|$.
 Consequently $D_\sigma R \in \mathcal{L}_{s,t}^{(e)}(X, \ell_q)$ for $1/s = 1/r + 1/q' - 1/p$.

Now we deal with the case $q = 1$ or $q = \infty$. Given r with $0 < r < q \leq \infty$, we choose r_0, r_1 and u such that $0 < r_0, r_1 < \infty$, $1 < u < \infty$, $1/r_0 + 1/u > 1/q$, $1/r_1 > 1/u$ and $1/r = 1/r_0 + 1/r_1$. Thus we split $\sigma = \mu \circ \tau$ with $\tau \in \ell_{r_1, t}$ and $\mu \in \ell_{r_0, \infty}$ and hence the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_q)$ is factorized with diagonal operators D_τ and D_μ as $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_u \xrightarrow{D_\mu} \ell_q$. Since $1/r_0 > \max(1/q - 1/u, 0)$, we take account of a result due to Carl [4] to conclude that $D_\mu \in \mathcal{L}_{s_0, \infty}^{(e)}(\ell_u, \ell_q)$ with $1/s_0 = 1/r_0 + 1/u - 1/q$. Applying the result of the preceding case to the operator $D_\tau \in \mathcal{B}(\ell_\infty, \ell_u)$ we have $D_\tau R \in \mathcal{L}_{s_1, t}^{(e)}(X, \ell_u)$ with $1/s_1 = 1/r_1 + 1/u' - 1/p$. The multiplication theorem for the entropy ideals assures us that $D_\sigma R = D_\mu D_\tau R \in \mathcal{L}_{s_0, \infty}^{(e)} \circ \mathcal{L}_{s_1, t}^{(e)}(X, \ell_q) \subseteq \mathcal{L}_{s, t}^{(e)}(X, \ell_q)$ whenever $1/s = 1/s_0 + 1/s_1 = 1/r + 1/q' - 1/p$. \square

Propositions 2 and 3 permit us to give a description of (r, w) -nuclear operators acting from a Banach space whose dual has weak type q into a Banach space of weak type p , $1 \leq p, q < 2$, in terms of their entropy numbers.

THEOREM 1. *Let X^* be of weak type q and Y of weak type p , $1 \leq p, q < 2$. Then $\mathcal{N}_{r, w}(X, Y) \subset \mathcal{L}_{s, w}^{(e)}(X, Y)$ provided that $0 < r < 1$, $0 < w \leq \infty$ and $1/s = 1 + 1/r - 1/p - 1/q$.*

Proof. We divide the proof into two steps.

Step 1. The first step is to verify the following assertion : If $R \in \mathcal{B}(X, \ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_r$, $0 < r < 1$, and $S \in \mathcal{B}(\ell_1, Y)$ then $SD_\sigma R \in \mathcal{L}_{s, r}^{(e)}(X, Y)$ with $1/s = 1 + 1/r - 1/p - 1/q$.

Since $1/r = 1/2r + 1/2r$, for $\sigma \in \ell_r$, we split $\sigma = \tau \circ \tau$ with $\tau \in \ell_{2r}$. Therefore the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ admits a factorization $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_2 \xrightarrow{D_\tau} \ell_1$, where D_τ is a diagonal operator induced by a sequence $\tau \in \ell_{2r}$. Since Y is of weak type p and $1/2r + 1/2 > 1$, we apply proposition 2 to deduce that $SD_\tau \in \mathcal{L}_{s_0, 2r}^{(e)}(\ell_2, Y)$ for $1/s_0 = 1/2r + 1/2 - 1/p$. As X^* is of weak type q and $0 < 2r < 2$, we invoke proposition 3 to produce $D_\tau R \in \mathcal{L}_{s_1, 2r}^{(e)}(X, \ell_2)$ with $1/s_1 = 1/2r + 1/2 -$

$1/q$. The multiplication theorem for the entropy ideals informs us that $SD_\sigma R = SD_\tau D_\tau R \in \mathcal{L}_{s_0, 2r}^{(e)} \circ \mathcal{L}_{s_1, 2r}^{(e)}(X, Y) \subseteq \mathcal{L}_{s, r}^{(e)}(X, Y) \subseteq \mathcal{L}_{s, \infty}^{(e)}(X, Y)$ with $1/s = 1/s_0 + 1/s_1 = 1 + 1/r - 1/p - 1/q$.

Step 2. We improve the result of the preceding step by real interpolation. Given r with $0 < r < 1$ we can find r_0, r_1 and θ such that $0 < r_0 < r_1 < 1$, $0 < \theta < 1$ and $1/r = (1 - \theta)/r_0 + \theta/r_1$. Let \mathcal{T} be the operator transforming every sequence σ into the composition operator $SD_\sigma R$. By Step 1, $\mathcal{T} : \ell_{r_i} \rightarrow \mathcal{L}_{s_i, \infty}^{(e)}(X, Y)$, where $1/s_i = 1 + 1/r_i - 1/p - 1/q$, $i = 0, 1$, are both bounded linear operators. Since $1/r = (1 - \theta)/r_0 + \theta/r_1$, we have $(\ell_{r_0}, \ell_{r_1})_{\theta, w} = \ell_{r, w}$ with the help of theorem 5.3.1. of [1]. The well-known interpolation formula concerning entropy ideals asserts that $(\mathcal{L}_{s_0, \infty}^{(e)}(X, Y), \mathcal{L}_{s_1, \infty}^{(e)}(X, Y))_{\theta, w} \subseteq \mathcal{L}_{s, w}^{(e)}(X, Y)$, where $1/s = (1 - \theta)/s_0 + \theta/s_1 = 1 + 1/r - 1/p - 1/q$. An appeal to the interpolation theorem [1] establishes that $\mathcal{T} : \ell_{r, w} \rightarrow \mathcal{L}_{s, w}^{(e)}(X, Y)$ is also bounded.

Now we select any $T \in \mathcal{N}_{r, w}(X, Y)$. It is known that T can be represented as $T = SD_\sigma R$, where $R \in \mathcal{B}(X, \ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r, w}$ and $S \in \mathcal{B}(\ell_1, Y)$. This leads us to have that $T = SD_\sigma R \in \mathcal{L}_{s, w}^{(e)}(X, Y)$. This proves the desired inclusion. \square

Naturally the question arises : Does the above theorem remain valid even when $p = 2$ or $q = 2$? This is answered by the theorems stated below. By applying a known fact concerning the relationship between entropy numbers and Kolmogorov numbers, together with estimates for the Kolmogorov numbers in terms of the Weyl numbers, we improve proposition 2 in Banach spaces of weak type 2.

PROPOSITION 4. *Let X be of weak type 2, $S \in \mathcal{B}(\ell_1, X)$ and let $D_\sigma \in \mathcal{B}(\ell_q, \ell_1)$ be a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r, t}$, where $1 \leq q \leq \infty$, $0 < r < \min(2, q')$ and $0 < t \leq \infty$. Then $SD_\sigma \in \mathcal{L}_{s, t}^{(e)}(\ell_q, X)$ with $1/s = 1/r + 1/q - 1/2$.*

Proof. First we shall show that if $\sigma \in \ell_r$ then $SD_\sigma \in \mathcal{L}_{s, \infty}^{(e)}(\ell_q, X)$, where $1/s = 1/r + 1/q - 1/2$. (*)

Given r with $0 < r < \min(2, q')$, we can pick r_1 and r_2 such that $0 < r_1 < 2$, $1/r_2 > \max(1/2 - 1/q, 0)$ and $1/r = 1/r_1 + 1/r_2$. Hence

for $\sigma \in \ell_r$, we split $\sigma = \tau \circ \mu$ with $\mu \in \ell_{r_2}$ and $\tau \in \ell_{r_1}$. Accordingly the operator $D_\sigma \in \mathcal{B}(\ell_q, \ell_1)$ can be factorized with diagonal operators D_μ and D_τ as $D_\sigma : \ell_q \xrightarrow{D_\mu} \ell_2 \xrightarrow{D_\tau} \ell_1$. Since $1/r_2 > \max(1/2 - 1/q, 0)$, we invoke a result of Carl [4] to deduce that $D_\mu \in \mathcal{L}_{s_2, r_2}^{(e)}(\ell_q, \ell_2) \subset \mathcal{L}_{s_2, \infty}^{(e)}(\ell_q, \ell_2)$ for $1/s_2 = 1/r_2 + 1/q - 1/2$. Using a result due to Carl [3] and making use of theorem 11.7.7. of [12] we get that $\|SD_\tau | \mathcal{L}_{r_1, \infty}^{(e)}\| \leq C_0 \|SD_\tau | \mathcal{L}_{r_1, \infty}^{(d)}\| = C_0 \|(SD_\tau)^* | \mathcal{L}_{r_1, \infty}^{(c)}\|$. Since X is of weak type 2, it follows that X is K -convex and hence X^* is K -convex. This enables us to use a result of Pajor and Tomczak-Jaegermann [11] to see that

$$\begin{aligned} \|(SD_\tau)^* | \mathcal{L}_{r_1, \infty}^{(c)}\| &\leq C_1 K(X^*)^{2/r_1} \|(SD_\tau)^* | \mathcal{L}_{r_1, \infty}^{(vr)}\| \\ &\leq C_1 K(X^*)^{2/r_1} \|SD_\tau | \mathcal{L}_{r_1, \infty}^{(v)}\|. \end{aligned}$$

Now we estimate the volume numbers in terms of the Weyl numbers. Let H be an n -dimensional subspace of ℓ_2 . By setting $E = \text{ran}(SD_\tau i_H)$, where $i_H : H \rightarrow \ell_2$ is the natural injection, we have $\dim E \leq n$. Then Lewis' theorem [14] asserts that there is an isomorphism $u : \ell_2^n \rightarrow E$ and an operator $v : X \rightarrow \ell_2^n$ such that $v|_E = u^{-1}$ and $\ell(u) = \ell^*(v) \leq n^{1/2}$. Using Geiss' theorem [9] and a result due to Pajor and Tomczak-Jaegermann [14], together with the multiplicativity of the volume numbers, we obtain the following :

$$\begin{aligned} v_n(SD_\tau i_H) &= v_n(uvSD_\tau i_H) \leq v_n(u) v_n(vSD_\tau i_H) \\ &\leq 4 e_n(u) \left(\prod_{k=1}^n a_k(vSD_\tau i_H) \right)^{1/n} \\ &\leq 4 C_2 n^{-1/2} \ell(u) n^{-(1/r_1+1/2)} \sup_k k^{1/r_1+1/2} x_k(v^*(SD_\tau i_H)^*) \\ &\leq 4 C_3 n^{-(1/r_1+1/2)} \|v | \mathcal{L}_{2, \infty}^{(a)}\| \cdot \|(SD_\tau)^* | \mathcal{L}_{r_1, \infty}^{(x)}\| \\ &\leq 4 C_3 n^{-(1/r_1+1/2)} wT_2(X) \ell^*(v) \|(SD_\tau)^* | \mathcal{L}_{r_1, \infty}^{(x)}\| \\ &\leq 4 C_3 n^{-1/r_1} wT_2(X) \|(SD_\tau)^* | \mathcal{L}_{r_1, \infty}^{(x)}\| \end{aligned}$$

This gives that $\|SD_\tau | \mathcal{L}_{r_1, \infty}^{(v)}\| \leq 4 C_3 wT_2(X) \|(SD_\tau)^* | \mathcal{L}_{r_1, \infty}^{(x)}\|$. From a result of Lubitz [13], we know that $\|(SD_\tau)^* | \mathcal{L}_{r_1, \infty}^{(x)}\| \leq C_4 \|S\| \cdot$

$\|\tau\|_{r_1, \infty} \leq C_4 \|S\| \cdot \|\tau\|_{r_1}$. Combining the above inequalities we arrive at $\|SD_\tau|_{\mathcal{L}_{r_1, \infty}^{(e)}}\| \leq CK(X^*)^{2/r_1} wT_2(X) \|S\| \cdot \|\tau\|_{r_1}$, that is $SD_\tau \in \mathcal{L}_{r_1, \infty}^{(e)}(\ell_2, X)$. An appeal to the multiplication theorem for the entropy ideals ensures that $SD_\sigma = SD_\tau D_\mu \in \mathcal{L}_{r_1, \infty}^{(e)} \circ \mathcal{L}_{s_2, \infty}^{(e)}(\ell_q, X) \subseteq \mathcal{L}_{s, \infty}^{(e)}(\ell_q, X)$ for $1/s = 1/r_1 + 1/s_2 = 1/r + 1/q - 1/2$, which proves (*).

Next we apply real interpolation to derive the required assertion. Given r with $0 < r < \min(2, q')$, we can select r_0, r_1 and θ such that $0 < r_0 < r_1 < \min(2, q')$, $0 < \theta < 1$ and $1/r = (1 - \theta)/r_0 + \theta/r_1$. Let \mathcal{T} be the operator assigning to every sequence σ the composition operator SD_σ . By (*), $\mathcal{T} : \ell_{r_i} \rightarrow \mathcal{L}_{s_i, \infty}^{(e)}(\ell_q, X)$, where $1/s_i = 1/r_i + 1/q - 1/2$, $i = 0, 1$, are both bounded linear operators. Since $1/r = (1 - \theta)/r_0 + \theta/r_1$, we have $(\ell_{r_0}, \ell_{r_1})_{\theta, t} = \ell_{r, t}$ in view of theorem 5.3.1. of [1]. The well-known interpolation formula concerning entropy ideals tells us that $(\mathcal{L}_{s_0, \infty}^{(e)}(\ell_q, X), \mathcal{L}_{s_1, \infty}^{(e)}(\ell_q, X))_{\theta, t} \subseteq \mathcal{L}_{s, t}^{(e)}(\ell_q, X)$, where $1/s = (1 - \theta)/s_0 + \theta/s_1 = 1/r + 1/q - 1/2$. Applying the interpolation theorem [1] we draw that $\mathcal{T} : \ell_{r, t} \rightarrow \mathcal{L}_{s, t}^{(e)}(\ell_q, X)$ is also bounded. This completes the proof of the proposition. \square

The following proposition shows a corresponding dual formulation for a Banach space whose dual has weak type 2.

PROPOSITION 5. *Let X^* be of weak type 2, $R \in \mathcal{B}(X, \ell_\infty)$ and let $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_q)$ be a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r, t}$, where $1 \leq q \leq \infty$, $0 < r < \min(2, q)$ and $0 < t \leq \infty$. Then $D_\sigma R \in \mathcal{L}_{s, t}^{(e)}(X, \ell_q)$ for $1/s = 1/r - 1/q + 1/2$.*

Proof. We proceed in the same way as in the proof of proposition 3. We first deal with the case $1 < q < \infty$. For $R \in \mathcal{B}(X, \ell_\infty)$, we define $R^*|_{\ell_1} = S \in \mathcal{B}(\ell_1, X^*)$. As X^* is of weak type 2 and $0 < r < \min(2, q)$, we summon up proposition 4 to conclude that $SD_\sigma \in \mathcal{L}_{s, t}^{(e)}(\ell_{q'}, X^*)$ with $1/s = 1/r + 1/q' - 1/2$. Since $\ell_{q'}$ is K -convex, we use a result due to Bourgain, Pajor, Szarek and Tomczak-Jaegermann [2] to obtain that there exists a constant $C \geq 0$ such that $\|D_\sigma R|_{\mathcal{L}_{s, t}^{(e)}}\| = \|D_\sigma S^*|_X|_{\mathcal{L}_{s, t}^{(e)}}\| \leq \|D_\sigma S^*|_{\mathcal{L}_{s, t}^{(e)}}\| \leq C \|SD_\sigma|_{\mathcal{L}_{s, t}^{(e)}}\|$. This implies $D_\sigma R \in \mathcal{L}_{s, t}^{(e)}(X, \ell_q)$ for $1/s = 1/r - 1/q + 1/2$.

Next we treat the case $q = 1$ or $q = \infty$. Given r with $0 < r < \min(2, q)$, we find r_0, r_1 and u such that $0 < r_0 < \infty$, $1 < u <$

∞ , $0 < r_1 < \min(2, u)$, $1/r_0 + 1/u > 1/q$ and $1/r = 1/r_0 + 1/r_1$. Therefore we split $\sigma = \mu \circ \tau$ with $\tau \in \ell_{r_1, t}$ and $\mu \in \ell_{r_0, \infty}$ and so the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_q)$ is factorized with diagonal operators D_τ and D_μ as $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_u \xrightarrow{D_\mu} \ell_q$. Since $1/r_0 > \max(1/q - 1/u, 0)$, we invoke a result of Carl [4] to infer that $D_\mu \in \mathcal{L}_{s_0, \infty}^{(e)}(\ell_u, \ell_q)$ with $1/s_0 = 1/r_0 + 1/u - 1/q$. Applying the result of the preceding case to the operator $D_\tau \in \mathcal{B}(\ell_\infty, \ell_u)$ we get $D_\tau R \in \mathcal{L}_{s_1, t}^{(e)}(X, \ell_u)$ with $1/s_1 = 1/r_1 - 1/u + 1/2$. Thanks to the multiplication theorem for the entropy ideals, we have $D_\sigma R = D_\mu D_\tau R \in \mathcal{L}_{s_0, \infty}^{(e)} \circ \mathcal{L}_{s_1, t}^{(e)}(X, \ell_q) \subseteq \mathcal{L}_{s, t}^{(e)}(X, \ell_q)$ for $1/s = 1/s_0 + 1/s_1 = 1/r - 1/q + 1/2$. \square

We are now in a position to extend theorem 1 to the case $p = 2$ or $q = 2$.

THEOREM 2. *Let X^* be of weak type 2 and Y of weak type p , $1 \leq p < 2$. Then $\mathcal{N}_{r, w}(X, Y) \subset \mathcal{L}_{s, w}^{(e)}(X, Y)$ provided that $0 < r < 1$, $0 < w \leq \infty$ and $1/s = 1/2 + 1/r - 1/p$.*

Proof. We divide the proof into two steps.

Step 1. The first step is to verify the following assertion : If $R \in \mathcal{B}(X, \ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_r$, $0 < r < 1$, and $S \in \mathcal{B}(\ell_1, Y)$ then $SD_\sigma R \in \mathcal{L}_{s, r}^{(e)}(X, Y)$ with $1/s = 1/2 + 1/r - 1/p$.

Since $1/r = 1/2r + 1/2r$, for $\sigma \in \ell_r$, we split $\sigma = \tau \circ \tau$ with $\tau \in \ell_{2r}$. Consequently the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ admits a factorization $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_2 \xrightarrow{D_\tau} \ell_1$, where D_τ is a diagonal operator generated by a sequence $\tau \in \ell_{2r}$. As Y is of weak type p and $1/2r + 1/2 > 1$, it follows from proposition 2 that $SD_\tau \in \mathcal{L}_{s_1, 2r}^{(e)}(\ell_2, Y)$ for $1/s_1 = 1/2r + 1/2 - 1/p$. Since X^* is of weak type 2 and $0 < 2r < \min(2, 2)$, we have $D_\tau R \in \mathcal{L}_{2r}^{(e)}(X, \ell_2)$ by way of proposition 5. Appealing to the multiplication theorem for the entropy ideals we derive that $SD_\sigma R = SD_\tau D_\tau R \in \mathcal{L}_{s_1, 2r}^{(e)} \circ \mathcal{L}_{2r}^{(e)}(X, Y) \subseteq \mathcal{L}_{s, r}^{(e)}(X, Y) \subseteq \mathcal{L}_{s, \infty}^{(e)}(X, Y)$ with $1/s = 1/s_1 + 1/2r = 1/2 + 1/r - 1/p$.

Step 2. We improve the result of the preceding step by real interpolation. Given r with $0 < r < 1$ we can select r_0, r_1 and θ such that $0 < r_0 < r_1 < 1$, $0 < \theta < 1$ and $1/r = (1 - \theta)/r_0 + \theta/r_1$. We

consider the operator \mathcal{T} transforming every sequence σ into the composition operator $SD_\sigma R$. By step 1, $\mathcal{T} : \ell_{r_i} \rightarrow \mathcal{L}_{s_i, \infty}^{(e)}(X, Y)$, where $1/s_i = 1/2 + 1/r_i - 1/p$, $i = 0, 1$, are both bounded linear operators. Applying the interpolation formulas given in the proof of theorem 1 we deduce that $\mathcal{T} : \ell_{r,w} \rightarrow \mathcal{L}_{s,w}^{(e)}(X, Y)$ is also bounded, where $1/s = (1 - \theta)/s_0 + \theta/s_1 = 1/2 + 1/r - 1/p$.

Now we take any $T \in \mathcal{N}_{r,w}(X, Y)$. It is known that T can be represented as $T = SD_\sigma R$, where $R \in \mathcal{B}(X, \ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_{r,w}$ and $S \in \mathcal{B}(\ell_1, Y)$. This allows us to have that $T = SD_\sigma R \in \mathcal{L}_{s,w}^{(e)}(X, Y)$. This proves the required inclusion. \square

THEOREM 3. *Let X^* be of weak type q , $1 \leq q < 2$, and Y of weak type 2. Then $\mathcal{N}_{r,w}(X, Y) \subset \mathcal{L}_{s,w}^{(e)}(X, Y)$ for $0 < r < 1$, $0 < w \leq \infty$ and $1/s = 1/2 + 1/r - 1/q$.*

Proof. Suppose that $R \in \mathcal{B}(X, \ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_r$, $0 < r < 1$, and $S \in \mathcal{B}(\ell_1, Y)$. Since $1/r = 1/2r + 1/2r$, for $\sigma \in \ell_r$, we split $\sigma = \tau \circ \tau$ with $\tau \in \ell_{2r}$. As a result the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ admits a factorization $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_2 \xrightarrow{D_\tau} \ell_1$, where D_τ is a diagonal operator induced by a sequence $\tau \in \ell_{2r}$. Since Y is of weak type 2 and $0 < 2r < \min(2, 2)$, we have $SD_\tau \in \mathcal{L}_{2r}^{(e)}(\ell_2, Y)$ with the aid of proposition 4. Since X^* is of weak type q and $0 < 2r < 2$, an appeal to proposition 3 reveals that $D_\tau R \in \mathcal{L}_{s_1, 2r}^{(e)}(X, \ell_2)$ for $1/s_1 = 1/2r + 1/2 - 1/q$. We apply the multiplication theorem for the entropy ideals to produce that $SD_\sigma R = SD_\tau D_\tau R \in \mathcal{L}_{2r}^{(e)} \circ \mathcal{L}_{s_1, 2r}^{(e)}(X, Y) \subseteq \mathcal{L}_{s,t}^{(e)}(X, Y) \subseteq \mathcal{L}_{s,\infty}^{(e)}(X, Y)$ with $1/s = 1/2r + 1/s_1 = 1/2 + 1/r - 1/q$. The required inclusion can be carried out by arguing exactly as in the second part of the proof of theorem 2. \square

THEOREM 4. *Let X^* be of weak type 2 and Y of weak type 2. Then $\mathcal{N}_{r,w}(X, Y) \subset \mathcal{L}_{r,w}^{(e)}(X, Y)$ where $0 < r < 1$ and $0 < w \leq \infty$.*

Proof. Assume that $R \in \mathcal{B}(X, \ell_\infty)$, $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ is a diagonal operator generated by a sequence $\sigma = (\sigma_i) \in \ell_r$, $0 < r < 1$, and $S \in \mathcal{B}(\ell_1, Y)$. Since $1/r = 1/2r + 1/2r$, for $\sigma \in \ell_r$, we split $\sigma = \tau \circ \tau$

with $\tau \in \ell_{2r}$. Accordingly the operator $D_\sigma \in \mathcal{B}(\ell_\infty, \ell_1)$ admits a factorization $D_\sigma : \ell_\infty \xrightarrow{D_\tau} \ell_2 \xrightarrow{D_\tau} \ell_1$, where D_τ is a diagonal operator generated by a sequence $\tau \in \ell_{2r}$. Since Y is of weak type 2 and $0 < 2r < \min(2, 2)$, it follows from proposition 4 that $SD_\tau \in \mathcal{L}_{2r}^{(e)}(\ell_2, Y)$. As X^* is of weak type 2 and $0 < 2r < \min(2, 2)$, proposition 5 steps in to assure that $D_\tau R \in \mathcal{L}_{2r}^{(e)}(X, \ell_2)$. Application of the multiplication theorem for the entropy ideals leads to $SD_\sigma R = SD_\tau D_\tau R \in \mathcal{L}_{2r}^{(e)} \circ \mathcal{L}_{2r}^{(e)}(X, Y) \subseteq \mathcal{L}_r^{(e)}(X, Y) \subseteq \mathcal{L}_{r, \infty}^{(e)}(X, Y)$. The remaining assertions are established by arguing exactly as in the second part of the proof of theorem 2. \square

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