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ON LEFT(RIGHT) SEMI-REGULAR *po*-SEMIGROUPS

S. K. Lee

Abstract. We give the characterization of left(right) semi-regular *po*-semigroups and V*e*-semigroups.

J. Calais([1]) proved that there are semigroups which are not regular, in which the left ideals are idempotent. The problem to describe the class of semigroups in which the left ideals are idempotent is due to S. Lajos([4]). He proved that normal semigroup is regular if and only if every ideal of S is idempotent. N. Kehayopulu([3]) proved that the same results for ordered semigroups and V*e*-semigroups are hold. Independently, for the way we work to apply the results of ordered semigroups or of *poe*($\vee e$)-semigroups based on ideal elements to semigroups -without order- we refer to [2].

In this paper, we give a characterization of the left(right) semi-regular *po*-semigroups and *poe*-semigroup, respectively. These are the improvements of Kehayopulu's results(see Corollaries 1 and 3).

A *po-semigroup*(: ordered semigroup) is an ordered set S at the same time a semigroup such that $a \leq b \Longrightarrow xa \leq xb$ and $ax \leq bx$ for all $x \in S$. A *poe-semigroup* is a *po*-semigroup with the greatest element e. A $\vee e$-semigroup is a *poe*-semigroup S at same time an upper semilattice satisfying the property $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all $a, b, c \in S$.

We denote $(H] = \{x \in S | x \leq h \text{ for some } h \in H\}$ for a subset H of a *po*-semigroup S. Then we can easily prove the followings;

(1) $A \subseteq (A]$ for any $A \subseteq S$.

(2) $(A] \subseteq (B]$ for $A \subseteq B \subseteq S$.

(3) $A = (A]$ for some types of ideal A.

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(4) $(A](B] \subseteq (AB]$ for $A, B \subseteq S$.

(5) For $A, B \subseteq S$, $(A \cap B] \neq (A] \cap (B]$, in general. In particular, if A and B are some types of ideal of S, then $(A \cap B] = (A] \cap (B]$.

A *po*-semigroup S is *left*(resp. *right*) *semi-regular* if $a \leq xaya$(resp. $a \leq ax'ay'$) for some $x, x', y, y' \in S$([5, 7]). A *po*-semigroup S is *quasi-completely regular* if $a \leq xaya$ or $a \leq ax'ay'$ for some $x, x', y, y' \in S$([6]).

A *poe*-semigroup S is is *left*(resp. *right*) *regular* if $a \leq eaea$(resp. $a \leq aeae$) for the greatest element $e \in S$. A *poe*-semigroup S is *quasi-completely regular* if $a \leq eaea$ or $a \leq aeae$ for the greatest element $e \in S$.

Let S be a *po*-semigroup. A nonempty subset A of S is called a *right*(resp. *left*) *ideal* of S if (1) $AS \subseteq A$(resp. $SA \subseteq A$), (2) $a \in A$ and $b \leq a$ for $b \in S \Longrightarrow b \in A$. A is called an *ideal* of S if it is a right and left ideal of S.

Let S be a *poe*-semigroup. An element a of S is called a *right*(resp. *left*) *ideal element* of S if $ae \leq a$(resp. $ea \leq a$). a is called an *ideal element* of S if it is a right and left ideal element of S.

We denote by $r(A)$(resp. $l(A)$) the right(resp. left) ideal of a *po*-semigroup S generated by A. Then $r(A) = (A \cup AS]$, $l(A) = (A \cup SA]$ in a *po*-semigroup S and $r(a) = a \vee ae$, $l(a) = a \vee ea$ in a *poe*-semigroup S.

A subset A of a *po*-semigroup S is *idempotent* if $A = (A^2]$. An element a of a *poe*-semigroup S is *idempotent* if $a = a^2$.

LEMMA 1. *Let S be a po-semigroup. Then the the followings are equivalent;*

(1) *S is left*(resp. *right*) *semi-regular.*

(2) *$a \in (SaSa]$*(resp. *$a \in aSaS]$*) *for all element $a \in S$.*

(3) *$A \subseteq (SASA]$*(resp. *$A \subseteq ASAS]$*) *for all subset A of S.*

THEOREM 1. *Let S be a po-semigroup. Then S is left*(resp. *right*) *semi-regular if and only if every left*(resp. *right*) *ideal of S is idempotent.*

Proof. $\Longrightarrow$. Let L be a left ideal of S. Since S is left regular, by Lemma 1

$$L \subseteq (SLSL] \subseteq (L^2] \subseteq (SL] \subseteq (L] = L.$$

Thus $(L^2] = L$, and so L is idempotent.

$\Longleftarrow$. Let A be any subset of S. Since $l(A)$ is the left ideal of S generated by A, by hypothesis we get

$$A \subseteq l(A) = (l(A)^2] \subseteq ((A \cup SA)(A \cup SA)] \\ \subseteq (A^2 \cup ASA \cup SA^2 \cup SASA].$$

Thus we have four cases:

(1) $A \subseteq A^2$. Then $A \subseteq A^2 \subseteq A^4 \subseteq SASA \subseteq (SASA]$.
(2) $A \subseteq ASA$. Then $A \subseteq ASA \subseteq AS(ASA) \subseteq (AS)ASA \subseteq SASA \subseteq (SASA]$.
(3) $A \subseteq SA^2$. Then $A \subseteq SA^2 \subseteq SA(SA^2) \subseteq SA(SA)A \subseteq SASA \subseteq (SASA]$.
(4) $A \subseteq SASA$. Then $A \subseteq SASA \subseteq (SASA]$.

In either case, we know that $A \subseteq (SASA]$. Hence S is left semi-regular. We can prove for the case right semi-regular by the similar method. □

If a *po*-semigroup S is regular, then it is left(right) semi-regular. And S is left(resp. right) semi-regular and $(SX] \subseteq (XS]$(resp. $(XS] \subseteq (XS]$) for a subset X of S, then it is regular. Therefore from Theorem 1, we get the following corollaries.

COROLLARY 1([3]). *Let S be a po-semigroup. If S is regular, then the left*(resp. *right) ideal of S are idempotent. "Conversely", let S be a po-semigroup such that $SX \subseteq XS$*(resp. *$(XS \subseteq SX)$ for all subset X of S, and the left* (resp. *right) ideal of S be idempotent. Then S is regular.*

COROLLARY 2. *Let S be a po-semigroup. Then S is quasi-completely regular if and only if every left or right ideal of S is idempotent.*

EXAMPLE 1(EXAMPLE 1 OF [3]). Let $S := \{a, b, c, d, e\}$ be a *po*-semigroup with Cayley table (Table 1) and Hasse diagram (Figure 1) as follows:

$\cdot$	a	b	c	d	e
a	b	b	b	b	b
b	b	b	b	b	b
c	c	c	c	c	c
d	c	c	c	d	d
e	c	c	c	d	e

Table 1

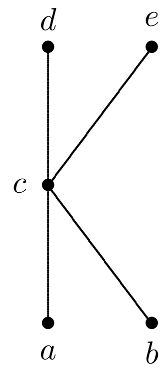

Figure 1

S is a left semi-regular *po*-semigroup. The left ideals of S are the sets $\{a,b,c\}$, $\{a,b,c,d\}$ and S, and they are idempotent.

But S is not right semi-regular. Indeed: $(aSaS] = (b] = \{b\}$ and $a \notin \{b\}$.

The right ideals of S are the sets $\{b\}, \{a,b\}, \{a,b,c\}$, $\{a,b,c,d\}$ and S. The right ideal $\{a,b\}$ is not idempotent. Indeed: $(\{a,b\}^2] = (b] = \{b\} \neq \{a,b\}$.

Theorem 2. *Let S be a $\vee e$-semigroup. Then S is left(*resp. *right) semi-regular if and only if every left(*resp. *right) ideal element of S is idempotent.*

Proof. $\Longrightarrow$. Let a be a left ideal element of S. Since S is left semi-regular, $a \leq eaea = (ea)(ea) \leq a^2 = aa \leq ea \leq a$. Thus $a^2 = a$, and so a is idempotent.

$\Longleftarrow$. Let a be any element of S. Since $l(a) = a \vee ea$ is the left ideal element of S generated by a, by hypothesis we get $a \leq l(a) = l(a)^2 = (a \vee ea)(a \vee ea) = a^2 \vee aea \vee ea^2 \vee eaea$. Thus we have four cases:

(1) $a \leq a^2$. Then $a \leq^2 \leq a^4 \leq eaea$.

(2) $a \leq aea$. Then $a \leq aea \leq (ae)(aea) \leq eaea$.

(3) $a \leq ea^2$. Then $a \leq ea^2 = eaa \leq ea(ea^2) \leq ea(ea)a \leq eaea$.

(4) $a \leq eaea$. Then it is obvious.

In either case, we know that $a \subseteq (eaea]$. Hence S is left semi-regular. We can prove for the case right semi-regular by the similar method. $\square$

If a $\vee e$-semigroup S is regular, then it is left(right) semi-regular. And S is left(resp. right) semi-regular and $ex \leq xe$(resp. $xe \leq xe$) for an element x of S, then it is regular. Therefore from Theorem 2, we get the following corollaries.

COROLLARY 3([3]). *Let S be a poe-semigroup. If S is regular, then the left*(resp. *right) ideal elements of S are idempotent. "Conversely", let S be a $\vee e$-semigroup such that $ex \le xe$*(resp. *$(xe \le ex)$ for all $x \in S$, and the left* (resp. *right) ideal elements of S be idempotent. Then S is regular.*

COROLLARY 4. *Let S be a $\vee e$-semigroup. Then S is quasi-completely regular if and only if every left or right ideal element of S is idempotent.*

EXAMPLE 2(EXAMPLE 7 OF [3]). Let $S := \{a, b, c, d, e\}$ be a *po*-semigroup with Cayley table (Table 2) and Hasse diagram (Figure 2) as follows:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	b
c	d	d	c	d	d
d	d	d	d	d	d
e	d	d	d	d	e

Table 2

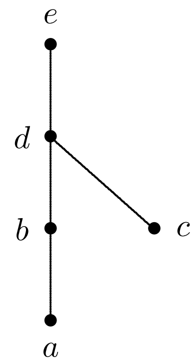

Figure 2

S is a left semi-regular $\vee e$-semigroup. The left ideal elements of S are the elements d and e, and they are idempotent. S is not right semi-regular. Indeed: $bebe = a^2 = a \not\geq b$. The right ideal elements of S are the elements a, b, d and e. The right ideal element b is not idempotent. Indeed: $b^2 = a$.

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Department of Mathematics
College of Education, Gyeongsang National University
Chinju 660-701, Korea.