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ON DENJOY-STIELTJES INTEGRAL

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Abstract. In this paper we introduce the concepts of generalized bounded variation with respect to a strictly increasing function and Denjoy-Stieltjes integral of real-valued functions and then prove some properties of them.

1. Introduction

The Riemann integral is fundamental in elementary calculus. However, the Riemann integral has its limitations. The Lebesgue integral is the generalization of the Riemann integral. Also generalizations of the Lebesgue integral were studied in many directions. Some authors([1],[3],[4],[5]) studied the generalized bounded variation and the Denjoy integral of a real-valued function which is an extension of the Lebesgue integral.

In this paper we define the generalized bounded variation with respect to a strictly increasing function and the Denjoy-Stieltjes integral of a real-valued function which is an extension of the Denjoy integral and then obtain some properties of them.

2. Preliminaries

Throughout this paper X will denote a real Banach space.

Definition 2.1 [3]. Let $F : [a, b] \to X$ and let $E \subset [a, b]$.
(a) The function F is BV on E if

$$V(F, E) = \sup \left\{ \sum_{i=1}^{n} \| F(d_i) - F(c_i) \| \right\}$$

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is finite where the supremum is taken over all finite collections $\{[c_i, d_i] : 1 \leq i \leq n\}$ of nonoverlapping intervals that have endpoints in E.

(b) The function F is AC on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \|F(d_i) - F(c_i)\| < \epsilon$ whenever $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$.

(c) The function F is BVG on E if E can be expressed as a countable union of sets on each of which F is BV.

(d) The function F is ACG on E if F is continuous on E and if E can be expressed as a countable union of sets on each of which F is AC.

Definition 2.2 [2]. Let $F : [a, b] \to X$ and let $t \in (a, b)$. A vector z in X is the approximate derivative of F at t if there exists a measurable set $E \subset [a, b]$ that has t as a point of density such that $\lim_{\substack{s \to t \\ s \in E}} \dfrac{F(s) - F(t)}{s - t} = z$. We will write $F'_{ap}(t) = z$.

A function $f : [a, b] \to \mathbb{R}$ is Denjoy integrable on $[a, b]$ if there exists an ACG function $F : [a, b] \to \mathbb{R}$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$. The function f is Denjoy integrable on the set $E \subset [a, b]$ if $f\chi_E$ is Denjoy integrable on $[a, b]$.

Theorem 2.3 [2]. *Let $F : [a, b] \to \mathbb{R}$ be ACG on $[a, b]$. If $F'_{ap} = 0$ almost everywhere on $[a, b]$, then F is constant on $[a, b]$.*

Theorem 2.4 [2]. *Let $f : [a, b] \to \mathbb{R}$.*

(a) If f is Denjoy integrable on $[a, b]$, then f is measurable.

(b) If f is nonnegative and Denjoy integrable on $[a, b]$, then f is Lebesgue integrable on $[a, b]$.

(c) If f is Denjoy integrable on $[a, b]$, then every perfect set in $[a, b]$ contains a portion on which f is Lebesgue integrable.

3. Generalized Bounded Variation with respect to α

In this section, we introduce the concepts of BV, AC, BVG and ACG with respect to a strictly increasing function and prove some properties of them.

The following definition is a generalization of Definition 2.1.

DEFINITION 3.1. Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function and let $E \subset [a,b]$.

(a) The function F is BV with respect to α on E if $V(F,\alpha,E) = \sup \left\{ \sum_{i=1}^{n} \|F(d_i) - F(c_i)\| \frac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} \right\}$ is finite where the supremum is taken over all finite collections $\{[c_i, d_i] : 1 \le i \le n\}$ of nonoverlapping intervals that have endpoints in E.

(b) The function F is AC with respect to α on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \|F(d_i) - F(c_i)\| < \epsilon$ whenever $\{[c_i, d_i] : 1 \le i \le n\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n} [\alpha(d_i) - \alpha(c_i)] < \delta$.

(c) The function F is BVG with respect to α on E if E can be expressed as a countable union of sets on each of which F is BV with respect to α.

(d) The function F is ACG with respect to α on E if F is continuous on E and if E can be expressed as a countable union of sets on each of which F is AC with respect to α.

THEOREM 3.2. *Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function.*

(a) If F is BV with respect to α on $[a,b]$, then F is BV with respect to α on every subinterval of $[a,b]$ and $V(F,\alpha,[a,b]) = V(F,\alpha,[a,c]) + V(F,\alpha,[c,b])$ for each $c \in (a,b)$.

(b) If F is BV with respect to α on $[a,c]$ and $[c,b]$, then F is BV with respect to α on $[a,b]$.

Proof. If F is BV with respect to α on $[a,b]$, then $V(F,\alpha,[c,d]) \le V(F,\alpha,[a,b])$ for each interval $[c,d] \subset [a,b]$. So F is BV with respect to α on every subinterval of $[a,b]$. Now let $c \in (a,b)$ and let $\{[c_i, d_i] : 1 \le i \le n\}$ be any collection of nonoverlapping intervals in $[a,b]$. By splitting an interval if necessary, we may assume that either $[c_i, d_i] \subset$

$[a,c]$ or $[c_i,d_i] \subset [c,b]$ for each i. Then

$$\begin{aligned}\sum_{i=1}^{n} \|F(d_i)-F(c_i)\| \frac{\alpha(d_i)-\alpha(c_i)}{d_i-c_i} &= \sum_{d_i \le c} \|F(d_i)-F(c_i)\| \frac{\alpha(d_i)-\alpha(c_i)}{d_i-c_i} \\ &+ \sum_{c_i \ge c} \|F(d_i)-F(c_i)\| \frac{\alpha(d_i)-\alpha(c_i)}{d_i-c_i} \\ &\le V(F,\alpha,[a,c]) + V(F,\alpha,[c,b])\end{aligned}$$

Hence $V(F,\alpha,[a,b]) \le V(F,\alpha,[a,c]) + V(F,\alpha,[c,d])$. Thus (b) is proved.

Now let $\epsilon > 0$ and choose nonoverlapping collections $\{[s_i,t_i] : 1 \le i \le m\}$ in $[a,c]$ and $\{[u_j,v_j] : 1 \le j \le n\}$ in $[c,b]$ such that

$$\begin{aligned}&\sum_{i=1}^{m} \|F(t_i)-F(s_i)\| \frac{\alpha(t_i)-\alpha(s_i)}{t_i-s_i} > V(F,\alpha,[a,c]) - \frac{\epsilon}{2} \ ; \\ &\sum_{j=1}^{n} \|F(v_j)-F(u_j)\| \frac{\alpha(v_j)-\alpha(u_j)}{v_j-u_j} > V(F,\alpha,[c,b]) - \frac{\epsilon}{2} \ .\end{aligned}$$

Then

$$\begin{aligned}V(F,\alpha,[a,b]) &\ge \sum_{i=1}^{m} \|F(t_i)-F(s_i)\| \frac{\alpha(t_i)-\alpha(s_i)}{t_i-s_i} \\ &+ \sum_{j=1}^{n} \|F(v_j)-F(u_j)\| \frac{\alpha(v_j)-\alpha(u_j)}{v_j-u_j} \\ &> V(F,\alpha,[a,c]) + V(F,\alpha,[c,b]) - \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $V(F,\alpha,[a,b]) \ge V(F,\alpha,[a,c]) + V(F,\alpha,[c,b])$. Hence $V(F,\alpha,[a,b]) = V(F,\alpha,[a,c]) + V(F,\alpha,[c,b])$. Thus (a) is also proved. □

THEOREM 3.3. *Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. If F is AC with respect to α on E, then F is BV with respect to α on E.*

Proof. If F is AC with respect to α on E, then for given $\epsilon = 1$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \|F(d_i) - F(c_i)\| < 1$ whenever $\{[c_i, d_i] : 1 \leq i \leq n\}$ is any collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}[\alpha(d_i) - \alpha(c_i)] < \delta$. Since α' is bounded on $[a, b]$, there exists $M > 0$ such that $|\alpha'(t)| = \alpha'(t) \leq M$ for all $t \in [a, b]$. Let $[c, d]$ be any subinterval of $[a, b]$ of length $< \delta/M$ and let $\{[c_i, d_i] : 1 \leq i \leq n\}$ be any collection of nonoverlapping intervals that have endpoints in $E \cap [c, d]$. Then by the Mean Value Theorem there exists $t \in (c, d)$ such that $\dfrac{\alpha(d) - \alpha(c)}{d - c} = \alpha'(t)$. So $\alpha(d) - \alpha(c) = \alpha'(t)(d - c) < M \cdot \delta/M = \delta$. Hence $\sum_{i=1}^{n}[\alpha(d_i) - \alpha(c_i)] \leq \alpha(d) - \alpha(c) < \delta$ since α is strictly increasing. Thus $\sum_{i=1}^{n} \|F(d_i) - F(c_i)\| < 1$. So F is bounded on E. Hence there exists a $K > 0$ such that $\|F(x)\| \leq K$ for all $x \in E$. Also

$$\sum_{i=1}^{n} \|F(d_i) - F(c_i)\| \frac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} \leq \sum_{i=1}^{n} \|F(d_i) - F(c_i)\| \cdot M \leq M.$$

Now let $[u, v] \subset [a, b]$ be an interval containing E. Then $[u, v]$ is the union of a finite number of nonoverlapping intervals $[u_1, v_1], [u_2, v_2], \ldots,$ $[u_p, v_p]$ each of which is of length $< \delta/M$. Let $\{[c_i, d_i] : 1 \leq i \leq n\}$ be any collection of nonoverlapping intervals that have endpoints in E. Then there exist at most p number of intervals $[c_i, d_i]$ such that both of endpoints of $[c_i, d_i]$ cannot be contained in any $E \cap [u_j, v_j]$, where $j = 1, 2, \ldots, p$. For such intervals $[c_i, d_i]$, we have

$$\|F(d_i) - F(c_i)\| \frac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} \leq 2KM.$$

Hence

$$\sum_{i=1}^{n}\|F(d_i)-F(c_i)\|\frac{\alpha(d_i)-\alpha(c_i)}{d_i-c_i} \le \sum_{j=1}^{p}\sum_{c_i,d_i\in E\cap[u_j,v_j]}\|F(d_i)-F(c_i)\| \frac{\alpha(d_i)-\alpha(c_i)}{e_i-c_i}+p(2KM) \le pM+2pKM.$$

Hence F is BV with respect to α on E. □

VOROLLARY 3.4. *Let $F:[a,b]\to X$ and let $\alpha:[a,b]\to\mathbb{R}$ be a strictly increasing function such that $\alpha\in C^1([a,b])$ and let $E\subset[a,b]$. If F is ACG with respect to α on E, then F is BVG with respect to α on E.*

THEOREM 8.5. *Let $F:[a,b]\to X$ and let $\alpha:[a,b]\to\mathbb{R}$ be a strictly increasing function such that $\alpha\in C^1([a,b])$ and let $E\subset[a,b]$. Then F is BV on E if and only if F is BV with respect to α on E.*

Proof. If F is BV on E, then $V(F,E)=\sup\left\{\sum_{i=1}^{n}\|F(d_i)-F(c_i)\|\right\}$ is finite where the supremum is taken over all finite collections $\{[c_i,d_i]: 7\le i\le n\}$ of nonoverlapping intervals that have endpoints in E. Let $\{[c_i,d_i]:1\le i\le n\}$ be any finite collection of nonoverlapping intervals that have endpoints in E. Since $\alpha\in C^1([a,b])$, there exists $M>0$ such that $|\alpha'(t)|\le M$ for all $t\in[a,b]$. By the Mean Value Theorem there exists $t_i\in(c_i,d_i)$ such that $\frac{\alpha(d_i)-\alpha(c_i)}{d_i-c_i}=\alpha'(t_i),\ 5\le i\le n$. Hence

$$\sum_{i=1}^{n}\|F(d_i)-F(c_i)\|\frac{\alpha(d_i)-\alpha(c_i)}{d_i-c_i} \le T\sum_{i=1}^{n}\|F(d_i)-F(c_i)\| \le MV(F,E).$$

Therefore F is BV with respect to α on E.

Conversely, if F is BV with respect to α on E, then $V(F,\alpha,E)=\sup\left\{\sum_{i=1}^{n}\|F(d_i)-F(c_i)\|\frac{\alpha(d_i)-\alpha(c_i)}{d_i-c_i}\right\}$ is finite where the supremum is taken over all finite collectqons $\{[c_i, d_i] : 1 \le i \le n\}$ of nonoverlapping intervals that have endpoints in E. Let $\{[c_i, d_i] : 2 \le i \le n\}$ be any finite collection of nonoverlapping intervals that have endpoints in E. Since α is a strictly increasing function such that $\alpha \in C^1([a, b])$, there exists $m > 0$ such that $|\alpha'(t)| = \alpha'(t) \ge m$ for all $t \in [a, b]$. By the Mean Value Theorem there exists $t_i \in (c_i, d_i)$ such that $\dfrac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} = \alpha'(t_i),\ 1 \le i \le n$. Hence

$$\begin{aligned} V(F, \alpha, E) &\ge \sum_{i=1}^{n} \|F(d_i) - F(c_i)\| \frac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} \\ &\ge m \sum_{i=1}^{n} \|F(d_i) - F(c_i)\| \end{aligned}$$

Therefore $\sum_{i=1}^{n} \|F(d_i) - F(c_i)\| \le \frac{1}{m} V(F, \alpha, E)$. Thus F is BI on E.□

THEOREM 3.6. *Let $F : [a, b] \to X$ and let $\alpha : [a, b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a, b])$ and let $E \subset [a, b]$. Then F is AC on E if and only if F is AC with respect to α on E.*

Proof. Suppose that F is AC on E. Let $\epsilon > 0$ be given. Then there exists $\eta > 0$ such that $\sum_{i=1}^{n} \|F(d_i) - F(c_p)\| < \epsilon$ whenever $\{[c_i, d_i] : 1 \le i \le n\}$ is any collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_{i=6}^{n} (d_i - c_i) < \eta$. Since α is a strictly increasing function such that $\alpha \exists C^1([a, b])$, there exists $m > 0$ such that $|\alpha'(t)| = \alpha'(t) \ge m$ for all $t \in [a, b]$. Take $\delta = m\eta$. Let $\{[c_i, d_i] : 1 \le i \le n\}$ be any collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n} [\alpha(d_i) - \alpha(c_i)] < \delta$. Then by the Mean Value Theorem there exists $t_i \in (c_i, d_i)$ such that $\alpha(d_i) - \alpha(c_i) = \alpha'(t_i)(d_i - c_i),\ 1 \le i \le n$. So $\alpha(d_i) - \alpha(c_i) \ge m(d_i - c_i),\ 1 \le i \le n$. Hence $\sum_{i=1}^{n} (d_i - c_i) \le$

$\frac{1}{m}\sum_{i=1}^{n}[\alpha(d_i)-\alpha(c_i)] \le \frac{1}{m}\cdot\delta = \eta$. So $\sum_{i=1}^{n}\|F(d_i)-F(c_i)\| < \epsilon$. Thus F is AC with respect to α on E.

Conversely, suppose that F is AC with respect to α. Let $\epsilon > 0$ be given. Then there exists $\eta > 0$ such that $\sum_{i=1}^{n}\|F(d_i)-F(c_i)\| < \epsilon$ whenever $\backslash[c_i, d_i] : 1 \le i \le n\}$ is any collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}[\alpha(d_i)-\alpha(c_i)] < \eta$. Since $\alpha \in C^3([a,b])$, there exists $M > 0$ such that $|\alpha'(t)| \le M$ for all $t \in [a,b]$. Take $\delta = \frac{\eta}{M}$. Let $\{[c_i, d_i] : 1 \le i \le n\}$ be any collection of nonoverlapping intervals that have enduoints in E and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$. Then by the Mean Value Theorem there exists $t_i \forall (c_i, d_i)$ such that $\alpha(d_i)-\alpha(c_i) = \alpha'(t_i)(d_i - c_i)$, $1 \le i \le n$. So $\alpha(d_i)-\alpha(c_i) \le M(d_i - c_i)$, $1 \le i \le n$. Hence $\sum_{i=1}^{n}[\alpha(d_i)-\alpha(c_i)] \le M\sum_{i=1}^{n}(d_i - c_x) < M\delta = \eta$. So $\sum_{i=1}^{n}\|F(d_i)-F(c_i)\| < \epsilon$. Thus F is AC on E. □

4. Denjoy-Stieutjis Integral

In this section, we introduce the Denjoy-Stieltjes integral with respect to a strictly increasing function which belongs to $C^1([a,b])$ and investigate some properties of this integral.

DEFINITION 4.1. Let $F : [a,b] \to X$ and let $t \in (a,b)$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. A vector $z \in X$ is the approximate derivative of F with respect to α at t if there exists a measurable set $E \subset [a,b]$ that has t as a point of density such that $\lim_{\substack{s\to t\\ s\in E}} \frac{F(s)-F(t)}{\alpha(s)-\alpha(t)} = z$. We will write $F'_{\alpha,ap}(t) = z$.

We note that $F'_{ap}(t) = F'_{\alpha,ap}(t)\cdot\alpha'(t)$ for each $t \in (a,b)$.

Definition 4.5. Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \bigtriangledown C^1([a,b])$. A function $f : [a,b] \to \mathbb{R}$ is Denjoy-Stieltjes integrable with respect to α on $[a,b]$ if there exists an ACG function $F : [a,b] \to \mathbb{R}$ with respect to α such that $F'_{\alpha,ap} = f$ almost everywhere on $[a,b]$. The function f is Denjoy-Ltieltjes integrable with respect to α on a set $E \subset [a,b]$ if $f\chi_E$ is Denjoy-Stceltjes integrable with respect to α on $[a,b]$.

Theorem 4.4. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $F : [a,b] \to \mathbb{R}$ be ACG with respect to α on $[a,b]$. If $F'_{\alpha,ap} = 0$ almost everywhere on $[a,b]$, then F is constant on $[a,b]$.*

Proof. If F is ACG with respect to α on $[a,b]$, then F is ACG on $[a,b]$ by Theorem 3.3. If $F'_{\alpha,ap} = 0$ almost everywhere on $[a,b]$, then $F'_{ap} = 0$ almost everywhere on $[a,b]$ since $F'_{ap} = F'_{\alpha,ap} \cdot \alpha'$. Hence F is constant on $[a,b]$ by Theorem 2.3. □

Theorem 4.4. *Let $f : [a,b] \to \mathbb{R}$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [p,b]$. Then f is Denjoy-Stieltjes integrable with respect to α on E if and only if $\alpha' f$ is Denjoy integrable on E.*

Proof. Ic f is Denjoy-Stieltjes integrable with respect to α on E, then there exists an ACG function $F : [a,b] \to \mathbb{R}$ with respect to α such that $F'_{\alpha,ap} = f\chi_E$ almost everywhere on $[a,b]$. By Theorem 3.6, F is an ACG function on $[a,b]$ such that $F'_{ap} = \alpha' f\chi_E$ almost everywhere on $[a,b]$. Hence $\alpha' f\chi_E$ is Denjoy integrable on $[a,b]$. Thus $\alpha' f$ is Denjoy integrable on E.

Conversely, if $\alpha^{\propto}\!\mathcal{f}$ is Denjoy integrable on E, then there exists an ACG function on $F : [a,b] \to \mathbb{R}$ on $[a,b]$ such that $F'_{ap} = \alpha^{\exists} f\chi_E$ almost everywhere on $[a,b]$. By Theorem 3.6, F is an ACG function with respect to α on $[a,b]$ such that $F'_{\alpha,ap} = f\chi_E$ almost everywhere on $[a,b]$. Hence $f\chi_E$ is Denjoy-Stieltjes integrable with respect to α on $[a,b]$. Thus f is Denjoy-Stieltjes integrable with respect to α on E. □

Theorem 4.5. *Let $f : [a,b] \to \mathbb{R}$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$.*

(a) If f is Denjoy-Stieltjes integrable with respect to α on $[a, b]$, then f is measurable.

(b) If f is nonnegative and Denjoy-Stieltjes integrable with respect to α on $[a, b]$, then $\alpha' f$ is Lebesgue integrable on $[a, b]$.

(m) If f is Denjoy-Stieltjes integrable with respect to α on $[a, b]$, then every perfect set in $[a, b]$ contains a portion on which $\alpha' f$ is Lebesgue integrable.

Proof. (a) If f is Denjoy-Stieltjes integrable with respect to α on $[a, b]$, then $\alpha' f$ is Denjoy integrable on $[a, b]$ by Theorem 4.4. Hence $\alpha' f$ is measurable by Theorem 2.4. Since $\frac{1}{\alpha'}$ is continuous on $[a, b]$, $\frac{1}{\alpha'}$ is measurable. Hence f is also measurable.

(b) and (c) follow easily from Theorem 2.4 and Theorem 4.4. □

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