Kangweon-Kyungki Math. Jour. **9** (2001), No. 2, pp. 123–128

THE CONSTRUCTION OF RELATIVE F-REGULAR RELATIONS

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Abstract. Given a homomorphism $\Pi : X \longrightarrow Y$, with Y minimal, we will introduce the concept of a relative (to Π) F-regular relation which generalize the notions of F-proximality, F-regularity and relative F-proximality, and will study its properties.

1. Introduction

The concepts of proximality and regularity have proved to be very fruitful for topological dynamics, giving rise to a rather extensive theory. H.S. Song [6] introduced the concept of a F-regular flow which is a slight generalization of that of a F-proximal flow. In this paper, we will introduce the concept of a relative (to Π) F-regular relation which generalize the notions of F-proximality, F-regularity and relative F-proximality, and will study its properties.

2. Preliminaries

In this paper, let T be an arbitrary, but a fixed topological group and we consider a flow (X, T) with compact Hausdorff space X. The *enveloping semigroup* $E(X)$ of (X, T) is the closure of $\{t : x \mapsto xt \mid t \in T\}$ in X^X.

A pair of points $(x, y), x, y \in X$ is said to be *proximal* if $xp = yp$ for some $p \in E(X)$. The proximal pairs is denoted by $P(X, T)$.

Received July 18, 2001.

2000 Mathematics Subject Classification: 54H20.

Key words and phrases: F-regular, relatively F-proximal, relatively F-regular, relative F-regular relations.

The present research has been conducted by the Research Grant of Kwangwoon University in 2000 .

We denote the endomorphisms of (X,T) by $H(X)$ and the automorphisms of (X,T) by $A(X)$. If $\phi \in H(X)$, we use the notation $\phi \in H_1(X)$ to denote $\phi\,|_M \in H(M)$ for any minimal subset M of (X,T). Similarly, if $\phi \in A(X)$, we use the notation $\phi \in A_1(X)$ to denote $\phi\,|_M \in A(M)$ for any minimal subset M of (X,T).

A pair of points $(x,y),\ x,y \in X$ is said to be *regular* provided that $(\phi(x),y) \in P(X,T)$ for some $\phi \in H_1(X)$. The regular pairs is denoted by $R(X,T)$.

For a flow (X,T), we define the first prolongation set and the first prolongation limit set of x in X respectively, by

$$D(x) = \{y \mid\ x_i t_i \to y \text{ for some } x_i \to x,\ t_i \in T\},$$
$$J(x) = \{y \mid\ x_i t_i \to y \text{ for some } x_i \to x,\ t_i \to \infty\},$$

where $t_i \to \infty$ means that the net $\{t_i\}$ is ultimately outside of each compact subset of T.

A point $x \in X$ is said to *have property M* if whenever there are nets $\{x_i\}$, $\{y_i\}$ in X and a net $\{t_i\}$ in T such that $x_i \to x, y_i \to x$ and the net $\{x_i t_i\}$ is convergent, then the net $\{y_i t_i\}$ is also convergent.

A flow (X,T) is said to be *T-weakly equicontinuous* if $J(x,x) \subset \Delta_X$ and x has property M, for every $x \in X$.

A pair of points $(x,x'),\ x,x' \in X$ is said to be*F-proximal* if $D(x,x') \cap \Delta_X \neq \emptyset$. Equivalently, $(x,x'),\ x,x' \in X$ is said to be *F-proximal* if there exist nets, $\{x_i\}$ and $\{x_i{'}\}$ in X, and a net $\{t_i\}$ in T such that $x_i \to x$, $x'_i \to x'$, and $\lim x_i t_i = \lim x'_i t_i$. The F-proximal pairs is denoted by $FP(X,T)$.

A pair of points $(x,x'),\ x,x' \in X$ is said to be *F-regular* provided that $(\phi(x),x') \in FP(X,T)$ for some $\phi \in H_1(X)$. The F-regular pairs is denoted by $FR(X,T)$.

Note that $P(X,T) \subset FP(X,T) \subset FR(X,T)$ and $P(X,T) \subset R(X,T)$ $\subset FR(X,T)$.

Given a homomorphism $\Pi : X \longrightarrow Y$, with Y minimal, we suppose $y \in Y$. Then $(X^{\Pi^{-1}(y)},T)$ is a compact Hausdorff flow. We define $z_y \in X^{\Pi^{-1}(y)}$ by $z_y(x) = x$ for all $x \in \Pi^{-1}(y)$ and let $E(\Pi,y)$ be the orbit closure of z_y. Then $(E(\Pi,y),T)$ is a compact Hausdorff subflow of $(X^{\Pi^{-1}(y)},T)$ and $E(X)$ is an enveloping semigroup for $E(\Pi,y)$. However, $E(\Pi,y)$ has no semigroup structure. Note that if Y is a singleton $\{y\}$, $E(\Pi,y)$ is just the $E(X)$, considered as a flow.

Now we define various fundamental notions as follows.

$$P_\Pi = \{(x,x') \in X \times X \mid \Pi(x) = \Pi(x'),\ (x,x') \in P(X,T)\},$$

$$P_\Pi(y) = \{(x,x') \in X \times X \mid \Pi(x) = \Pi(x') = y,\ (x,x') \in P(X,T)\},$$

$$R_\Pi = \{(x,x') \in X \times X \mid \Pi(x) = \Pi(x'), (\phi(x),x') \in P(X,T) \text{ for some } \phi \in H_1(X)\},$$

$$R_\Pi(y) = \{(x,x') \in X \times X \mid \Pi(x) = \Pi(x') = y, (\phi(x),x') \in P(X,T) \text{ for some } \phi \in H_1(X)\}.$$

3. Relative F-regular relations

In this section we will work with a fixed homomorphism $\Pi : X \longrightarrow Y$, where Y is minimal.

Definition 3.1. A pair of points (x,x'), $x,x' \in X$ is *relatively F-proximal* or belongs to the *relative (to* Π*) F-proximal relation* if there exist nets $\{x_i\}$, $\{x_i{}'\}$ in X and a net $\{t_i\}$ in T such that $\Pi(x_i) = \Pi(x_i')$ for each i, $x_i \to x$, $x_i' \to x'$ and $\lim x_i t_i = \lim x_i' t_i$. The relative F-proximal relation is denoted by FP_Π.

Note that if $(x,x') \in FP_\Pi$, then $\Pi(x) = \Pi(x')$. Given $y \in Y$, define the set $FP_\Pi(y) = \{(x,x') \in X \times X \mid \Pi(x) = \Pi(x') = y,\ (x,x') \in FP(X,T)\}$.

Definition 3.2. The *relative F-regular relation*, denoted by FR_Π, is the set

$\{(x,x') \in X \times X \mid \Pi(x) = \Pi(x'), (\phi(x),x') \in FP(X,T)$ for some $\phi \in H_1(X)\}$.

Given $y \in Y$, $FR_\Pi(y) = \{(x,x') \in X \times X \mid \Pi(x) = \Pi(x') = y, (\phi(x),x') \in FP(X,T)$ for some $\phi \in H_1(X)\}$.

Remark 3.3. (1) $P_\Pi \subset FP_\Pi \subset FR_\Pi$.

(2) $P_\Pi \subset R_\Pi \subset FR_\Pi$.

Remark 3.4. Note that $P_\Pi = P_\Pi(1) = P(X,T)$, $R_\Pi = R_\Pi(1) = R(X,T)$, $FP_\Pi = FP_\Pi(1) = FP(X,T)$, and $FR_\Pi = FR_\Pi(1) = FR(X,T)$ when applied to the unique homomorphism $\Phi : X \longrightarrow 1$, where 1 is the one-point flow.

In [6], Song studied the F-regular relations and proved the following theorem :

THEOREM 3.5. *(1) If (X,T) is T-weakly equicontinuous, then $FP(X,T) = P(X,T)$ and $FR(X,T) = R(X,T)$.*

(2) If $(x,x') \in FP(X,T)$ and $\phi \in H(X)$, then $(\phi(x),\phi(x')) \in FP(X,T)$.

(3) If $(x,x') \in FP(X,T)$ and $\sigma : (X,T) \longrightarrow (Y,T)$ is a homomorphism, then $(\sigma(x),\sigma(x')) \in FP(Y,T)$.

(4) Let $H_1(X)$ be algebraically transitive (that is, if $x,x' \in X$, there is a $\eta \in H_1(X)$ with $\eta(x) = x'$) and let $(x,x') \in FR(X,T)$ and $\phi \in H_1(X)$. Then $(\phi(x),\phi(x')) \in FR(X,T)$.

(5) Let $\sigma : (X,T) \longrightarrow (Y,T)$ be an epimorphism, and assume that $H_1(Y)$ is algebraically transitive. If (X,T) is F-regular, then (Y,T) is F-regular.

COROLLARY 3.6. *If (X,T) is T-weakly equicontinuous, then $FP_\Pi = P_\Pi$ and $FR_\Pi = R_\Pi$.*

Proof. This follows from Definition 3.1 and Theorem 3.5.1. □

COROLLARY 3.7. *(1) If $(x,x') \in P(X,T)$, then $(\Pi(x),\Pi(x')) \in P(Y,T)$.*

(2) If $(x,x') \in FP(X,T)$, then $(\Pi(x),\Pi(x')) \in FP(Y,T)$.

(3) If $(x,x') \in R(X,T)$ and $H_1(Y)$ is algebraically transitive, then $(\Pi(x),\Pi(x')) \in R(Y,T)$.

(4) If $(x,x') \in FR(X,T)$ and $H_1(Y)$ is algebraically transitive, then $(\Pi(x),\Pi(x')) \in FR(Y,T)$.

THEOREM 3.8. *Let $H_1(X)$ be algebraically transitive. If $\Pi : X \longrightarrow Y$ is an one-to-one extension of F-regular flow, then (X,T) is also F-regular.*

Proof. For any $x_1,x_2 \in X$, there exist $y_1,y_2 \in Y$ such that $\Pi(x_1) = y_1, \Pi(x_2) = y_2$. Since (Y,T) is F-regular, there exists a $\psi \in H_1(Y)$ such that $(\psi(y_1),y_2) \in FP(Y,T)$. Since an almost one-to-one extension of a minimal F-proximal flow is F-proximal, we have $(\Pi^{-1}(\psi(y_1)),\Pi^{-1}(y_2)) \in FP(X,T)$ (see Proposition 2.7 in [4]). But since $H_1(X)$ is algebraically transitive, there is a $\zeta \in H_1(X)$ with $\zeta(x_1) = \Pi^{-1}(\psi(y_1))$, it follows that $(x_1,x_2) \in FR(X,T)$. We thus have (X,T) is F-regular. □

LEMMA 3.9. [5] *Let $y \in Y$. Then $P_\Pi(y)$ is an equivalence relation if and only if $E(\Pi,y)$ contains just one minimal set.*

THEOREM 3.10. *If $H_1(X)$ is a group, then $R_\Pi(y)$ is a reflexive and symmetric relation on $\Pi^{-1}(y)$.*

Proof. For any $x \in \Pi^{-1}(y)$, we have $(x, x) \in P_\Pi(y) \subset R_\Pi(y)$. To show that $R_\Pi(y)$ is symmetric, let $(x, x') \in R_\Pi(y)$. Then $\Pi(x) = \Pi(x') = y$ and there exists a $\phi \in H_1(X)$ such that $(\phi(x), x') \in P(X, T)$. Hence $(x', \phi(x)) \in P(X, T)$. But since $\phi^{-1} \in H_1(X)$, it follows that $(\phi^{-1}(x'), x) \in P(X, T)$. Hence $(x', x) \in R_\Pi(y)$. Therefore $R_\Pi(y)$ is symmetric. □

It is well-known that if (X, T) is regular minimal, then every endomorphism of (X, T) is an automorphism. Therefore we have

COROLLARY 3.11. *If (X, T) is regular minimal, then $R_\Pi(y)$ is a reflexive and symmetric relation on $\Pi^{-1}(y)$.*

THEOREM 3.12. *Let (X, T) be regular minimal. If $E(\Pi, y)$ contains just one minimal set, then $R_\Pi(y)$ is an equivalence relation on $\Pi^{-1}(y)$.*

Proof. It suffices to show that $R_\Pi(y)$ is transitive. Suppose $E(\Pi, y)$ contains just one minimal set. Let $x, x'$and x'' be in X such that $(x, x') \in R_\Pi(y)$ and $(x', x'') \in R_\Pi(y)$. Then $\Pi(x) = \Pi(x') = \Pi(x'') = y$ and there exist $\phi, \psi \in H_1(X)$ such that $(\phi(x), x'), (\psi(x'), x'') \in P(X, T)$. Hence $(\psi\phi(x), \psi(x')), (\psi(x'), x'') \in P(X, T)$. Lemma 3.9 shows that $(\psi\phi(x), x'') \in P(X, T)$. Therefore $R_\Pi(y)$ is transitive. □

THEOREM 3.13. *Let (X, T) be regular minimal and let $y \in Y$. The following statements are equivalent :*

(a) $R_\Pi(y)$ is an equivalence relation on $\Pi^{-1}(y)$.

(b) Let u be an idempotent with $yu = y$. Then $(xu, x'u) \in R_\Pi(y)$ for every $(x, x') \in R_\Pi(y)$.

(c) Let u be an idempotent with $yu = y$ and v be an equivalent idempotent with u. Then $(xu, xv) \in R_\Pi(y)$ for every $x \in \Pi^{-1}(y)$.

Proof. (a) implies (b). Let $(x, x') \in R_\Pi(y)$ and let u be an idempotent with $yu = y$. Note that $(x, xu) \in R_\Pi(y)$ for all $x \in \Pi^{-1}(y)$. Since $(x, xu) \in R_\Pi(y)$ and $(x', x'u) \in R_\Pi(y)$, and $R_\Pi(y)$ is an equivalence relation on $\Pi^{-1}(y)$, it follows that $(xu, x'u) \in R_\Pi(y)$.

(b) implies (c). Note that $yv = y(u)v = y(uv) = yu = y$. Because $(xu, x) \in P_\Pi(y)$ for all $x \in \Pi^{-1}(y)$ and $P_\Pi(y) \subset R_\Pi(y)$, we have that $(xu, x) \in R_\Pi(y)$. Therefore $(xuv, xv) = (xu, xv) \in R_\Pi(y)$ by the condition (b).

(c) implies (a). It suffices to show that $R_\Pi(y)$ is transitive. Let $(x, x') \in R_\Pi(y)$ and $(x', x'') \in R_\Pi(y)$. Then $\Pi(x) = \Pi(x') = \Pi(x'') = y$ and there exist $\phi, \psi \in H_1(X)$ such that $(\phi(x), x'), (\psi(x'), x'') \in P(X, T)$.

Therefore there are minimal right ideals I and K in $E(X)$ such that $\phi(x)p = x'p$ and $\psi(x')q = x''q$ for all $p \in I$, $q \in K$. Now let u be an idempotent in I with $yu = y$ and v be an equivalent idempotent with u in K. Then $\phi(x)u = x'u$ and $\psi(x')v = x''v$. Thus we have from the condition (c) that $(x'u, x'v) \in R_\Pi(y)$ and $(x''u, x''v) \in R_\Pi(y)$. Hence $(\eta(x'u), x'v), (\zeta(x''u), x''v) \in P(X,T)$ for some $\eta, \zeta \in H_1(X)$. But since $(\eta(x'u), x'v)$ and $(\zeta(x''u), x''v)$ are almost periodic points, we have that $\eta(x'u) = x'v$ and $\zeta(x''u) = x''v$. It follows that $\zeta(x''u) = x''v = \psi(x')v = \psi(\eta(x'u)) = \psi(\eta(\phi(x)u)) = \psi\eta\phi(x)u$. Therefore $\zeta^{-1}\psi\eta\phi(x)u = x''$ and hence $(x, x'') \in R_\Pi(y)$. □

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