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MAPPING CLASS GROUP OPERAD

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Abstract. We construct an operad which is called the mapping class group operad. Its structure map is induced by the attachings of surfaces. The similar operad was constructed by Tillmann in order prove that Γ_∞^+ is an infinite loop space. Our operad is rather simpler and easier to deal with.

1. Introduction

The theory of operads was introduced by homotopy theorists in the early seventies in order to understand and detect iterated and in particular infinite loop spaces. The little n-cube operad $\mathcal{C}_n$ introduced by Boardman and Vogt([2]) has been playing a key role in investigating the internal structure of iterated loop spaces.

It was announced by E. Miller([5]) that the classifying space of the collection of mapping class groups with one boundary component has the homotopy type of double loop space, and that he noticed the action of little 2-cube operad on it. Tillman([6]) proved that the space is really an infinite loop space by using a higher genus surface operad. In this paper we construct the operad which is simpler and more direct because we do not take any quotients on the level of categories. It is important to note that the structure maps of the mapping class group operad $\mathcal{S}$ induced by the attachings do not produce a surface of type $F_{0,2}$. We may take a single object S^1 in the category $\mathcal{S}_{0,1,1}$.

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2. Preliminaries

In this section we review some definitions and well-known results concerning mapping class groups, classifying space of categories, ribbon braid group and operads; for more details see ([1], [3], [4]).

Let $F_{g,k}$ be a compact connected orientable surface of genus g with k boundary components. Let $Diff^+(F_{g,k})$ be the group of orientation preserving diffeomorphisms of $F_{g,k}$ that fix the boundary pointwise, and let $Iso(F_{g,k})$ be the normal subgroup of diffeomorphisms which are isotopic to the identity relative to the boundary. The quotient group $\Gamma_{g,k} = Diff^+(F_{g,k})/Iso(F_{g,k})$ is called the *mapping class group* of the surface $F_{g,k}$. It is well-known that $\Gamma_{g,k}$ is isomorphic to the group of isotopy classes of those self-diffeomorphisms, and each connected component of $Diff^+(F_{g,k})$ is contractible so that $\Gamma_{g,k} = \pi_0 Diff^+(F_{g,k})$.

Definition 2.1. Let $\mathcal{C}$ be a small category. Then we can form a simplicial set $B_*\mathcal{C}$, which is called the *bar construction* (or *nerve*) of $\mathcal{C}$. Let $B_0\mathcal{C} = Obj(\mathcal{C})$. For $n \geq 1$, n-simplices $B_n\mathcal{C}$ is a set of all possible chains of morphisms of the form

$$A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} A_n, \quad A_i \in Obj(\mathcal{C}), \quad \alpha_j \in Mor(\mathcal{C}).$$

The i-th face map d_i deletes the i-th object and composes maps if necessary. The i-th degeneracy map s_i replaces A_i by $A_i \xrightarrow{id} A_i$. The *classifying space* $B\mathcal{C}$ of a category $\mathcal{C}$ is defined $B\mathcal{C} = |B_*\mathcal{C}|$. In particular, for a group G, we may regard G as a category with a single object $*$ and morphism $* \xrightarrow{g} *$ for all $g \in G$. Here the composition of morphisms is the multiplication. Then we have $BG = K(G, 1)$, the Eilenberg-MacLane space. The functor $B : \mathcal{C}at \to \mathcal{T}op$ is called the *classifying space functor.*

Let F, G be functors from $\mathcal{C}$ to $\mathcal{D}$. If there is a natural transformation $\eta : F \to G$ then η induces a homotopy $BF \overset{B\eta}{\simeq} BG$.

Lemma 2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then*

$$B(\mathcal{C} \times \mathcal{D}) \simeq B\mathcal{C} \times B\mathcal{D}.$$

Lemma 2.3. *Let $\mathcal{C}$ be a connected groupoid. Then $B\mathcal{C}$ is homotopy equivalent to $BHom(x, x)$ for each $x \in Obj(\mathcal{C})$.*

Proof. We first show that for pairs (x_0, y_0), (x_1, y_1) of objects,

$$Hom(x_0, y_0) \cong Hom(x_1, y_1).$$

Choose isomorphisms $f : x_1 \to x_0$, $g : y_0 \to y_1$. Define $\Phi : Hom(x_0, y_0) \to Hom(x_1, y_1)$ and $\Psi : Hom(x_1, y_1) \to Hom(x_0, y_0)$ as follows: $\Phi(\phi) = g \circ \phi \circ f$ for $\phi \in Hom(x_0, y_0)$, $\Psi(\psi) = g^{-1} \circ \psi \circ f^{-1}$ for $\psi \in Hom(x_1, y_1)$. Then $\Psi \circ \Phi(\phi) = \phi$ and $\Phi \circ \Psi(\psi) = \psi$.

Let $G = Hom(x_0, x_0)$ for a fixed object $x_0 \in \mathcal{C}$, and let $G \overset{i}{\hookrightarrow} \mathcal{C}$. Let $\phi(x, y) : Hom(x, y) \to G$ be an isomorphism. Now, we can define a functor $F : \mathcal{C} \to G$ as follows: $F(x) = x_0$ for $x \in Obj(\mathcal{C})$, $F(f) = \phi(x, y)(f)$ for $f : x \to y \in Hom(x, y)$. Then $F \circ i = id_G$ and we get a natural transformation $\eta : i \circ F \to id_{\mathcal{C}}$ such that the following diagram commutes

$$\begin{array}{ccc}
id_{\mathcal{C}}(x) = x & \xrightarrow{f} & id_{\mathcal{C}}(y) = y \\
\eta_x \Big\downarrow & & \Big\downarrow \eta_y \\
F(x) = x_0 & \xrightarrow{\phi(x,y)(f)} & F(y) = x_0 \quad .
\end{array}$$

Hence $B\mathcal{C} \simeq BHom(x_0, x_0)$. $\square$

Definition 2.4. (a) An *operad* $\mathcal{C}$ is a family of spaces $\mathcal{C}(j)$ for $j \geq 0$, such that the following conditions hold:

1. The space $\mathcal{C}(0)$ contains exactly one point $*$.
2. Continuous functions(called structure maps)

$$\gamma : \mathcal{C}(k) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k) \longrightarrow \mathcal{C}(j), j = \sum_{i=1}^{k} j_i$$

are given such that the associativity relation

$$\gamma(\gamma(c; d_1, \cdots, d_k)); e_1, \cdots, e_j) = \gamma(c; f_1, \cdots, f_k)$$

is satisfied, where $c \in \mathcal{C}(k), d_i \in \mathcal{C}(j_i), e_s \in \mathcal{C}(i_s)$, and

$$f_i = \begin{cases} \gamma(d_i, e_{j_1+\cdots+j_{i-1}+1}, \cdots, e_{j_1+\cdots+j_i}) & j_i \neq 0 \\ * & j_i = 0. \end{cases}$$

3. There is a distinguished element $1 \in \mathcal{C}(1)$ such that $\gamma(1; d) = d$ for any $d \in \mathcal{C}(j)$ and $\gamma(c; 1, \cdots, 1) = c$ for any $c \in \mathcal{C}(k)$.

4. Right actions of the symmetric groups Σ_j on the spaces $\mathcal{C}(j)$ are given, and the following equivariance relations hold:

$$\gamma(c\sigma; d_1, \cdots, d_k) = \gamma(c; d_{\sigma^{-1}(1)}, \cdots, d_{\sigma^{-1}(k)})\sigma(j_1, \cdots, j_k),$$
$$\gamma(c; d_1\tau_1, \cdots, d_k\tau_k) = \gamma(c; d_1, \cdots, d_k)(\tau_1 \oplus \cdots \oplus \tau_k),$$

where $\sigma(j_1, \cdots, j_k)$ is the permutation of j elements which is defined by partitioning the set of these elements into k blocks of $j_1, \cdots, j_k$ elements and acting on the blocks by the permutation σ; $\tau_1 \oplus \cdots \oplus \tau_k$ is the image of $(\tau_1, \cdots, \tau_k)$ under the natural embedding of the direct product $\Sigma_{j_1} \times \cdots \Sigma_{j_k}$ into Σ_j.

(b) A map of operads $f : \mathcal{C} \to \mathcal{D}$ is a family $f = \{f(j)\}$ of Σ_j-equivalent continuous maps $f(j) : \mathcal{C}(j) \to \mathcal{D}(j)$ which commute with γ's.

The operads we will consider in this paper are all constructed from families of discrete groups and groupoids. Let G be a group and H be a subgroup. Let $\mathcal{C}_H^G$ denote the category with the set of objects G/H, the left cosets of H, and morphism sets $\mathcal{C}_H^G(g_0H, g_1H) = \{g \in G | gg_0H = g_1H\} \simeq H$. An element h of which may be identified with left multiplication by $g_1hg_0^{-1}$. Then by Lemma 2.3, we have $B\mathcal{C}_H^G \simeq BH$.

Let (G_n, H_n) be a family of pair of groups with $G_n/H_n = \Sigma_n$. Assume there are wreath products $G_k \int G_n \xrightarrow{\omega} G_{kn}$ which restrict to the family of subgroups. If ω's satisfy the necessary associativity and identity conditions, the the disjoint union $\amalg_{n\geq 0} B\mathcal{C}_{H_n}^{G_n}$ forms an operad with free action of the symmetric groups whose structure maps are induced by ω.

Recall that the wreath product $G_k \int G_n$ is $G_k \times G_n^k = G_k \times G_n \times \cdots \times G_n$ as a set. Let $x = (c; g_1, \cdots, g_k)$ and $y = (d; h_1, \cdots, h_k)$ be in $G_k \int G_n$ with $c, d \in G_k$ and $g_i, h_j \in G_n$. Then the product xy is defined to be $(cd; g_{\sigma(1)}h_1, g_{\sigma(2)}h_2, \cdots, g_{\sigma(k)}h_k)$ where σ in Σ_k is given by $\sigma = p(d)$ for some map $p : G_k \to \Sigma_k$ (G_k acts on $\{1, \cdots, k\}$ as a permutation).

Example 2.5. The braid group operad $\mathcal{B}$ constructed from the family of pairs $(\beta_n, P\beta_n)$ where β_n is Artin's braid group on n strands and $P\beta_n$ is the pure braid group. The wreath product $\omega(g; g_1, \cdots, g_n)$ is obtained by replacing the i-th strand in g by the braid g_i as in Figure 1.

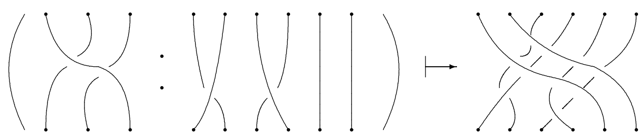

Figure 1.

Recall that a presentation for the ribbon braid group $R\beta_n$ is given by the generators s_i for $1 \le i \le n$ and the relations:

$$\begin{cases} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } 1 \le i \le n-2 \\ s_i s_j = s_j s_i \text{ for } |i-j| > 1 \\ s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1} \end{cases}$$

For example, the i-th twist element is expressed by

$$s_i^{-1} s_{i+1}^{-1} \cdots s_{n-1}^{-1} s_n s_{n-1} \cdots s_{i+1} s_i \quad \text{for } 1 \le i \le n-1.$$

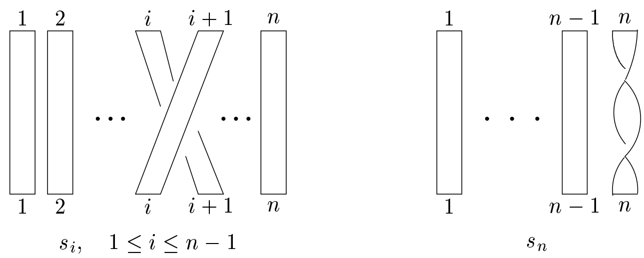

Figure 2.

EXAMPLE 2.6. The ribbon braid group operad $\mathcal{RB}$ constructed from the family of pairs $(R\beta_n, PR\beta_n)$. The strands are replaced by ribbons which might be twisted. The wreath product is defined just as for the braid groups. The symmetric operad Γ constructed from the pair (Σ_n, e); this is an E_∞-operad in the sense of [3].

3. Mapping Class Group Operad $\mathcal{S}$

For the collection of mapping class groups $\Gamma_{g,n+1}$ of genus g and with $n+1$ boundary components, let $S_n = \amalg_{g \ge 0} B\Gamma_{g,n+1}$. We like to construct directly an operad which may give us an information of loop space structures of mapping class groups.

Let $F_{g,n+1}$ be a compact oriented surface of genus g with $n+1$ boundary components. We may think that one of the boundary components of $F_{g,n+1}$ is marked and the remaining n boundary components are free. We may define wreath products

$$\Gamma_{g,k+1} \int \Gamma_{h,n+1} \longrightarrow \Gamma_{g+kh,kn+1}$$

by attaching the marked boundary components of k copies of the surfaces $F_{h,n+1}$ to the k free boundary components of $F_{g,k+1}$. We call this the **attachings**. This might give an operad structure to $S = \amalg_{n\geq 0} S_n$, which may not play an interesting role because it has trivial actions of the symmetric groups Σ_n. Moreover, there is a technical problem in identifying the surfaces with the same genus and the same number of boundary components generated by the above attachings.

We now define groupoids which give rise to an operad which has the same homotopy type as S, and may play more interesting role because there is a natural map from this operad to the symmetric operad Γ.

We first consider the normal extension of $\Gamma_{g,n+1}$ by Σ_n:

$$\Gamma_{g,n+1} \longrightarrow \Gamma_{g,n,1} \longrightarrow \Sigma_n.$$

If we let $F_{g,n,1}$ be a fixed surface with one marked boundary component and n ordered boundary components, we may regard $\Gamma_{g,n,1}$ be the group of isotopy classes of self-diffeomorphisms of $F_{g,n,1}$ which permute the n boundary components.

The construction of mapping class group operad We now construct an operad which is derived from some groupoids and whose structure maps are induced by the attachings.

Definition 3.1. Let F denote a surface of type $F_{g,n,1}$ which is obtained by attaching **block surfaces** as follows:

(A) For $(g, n) \neq (0, 1)$, a surface F is obtained by attaching a pair of pants $P = F_{0,3}$, a torus $T = F_{1,2}$ with two boundary components and a disk D each of which has one marked boundary component.

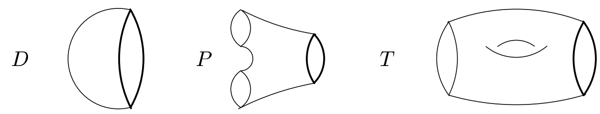

Figure 3.

We should attach a marked boundary component to one of free boundary components. We finally give an ordering to n free boundary components.

Let $\mathcal{S}_{g,n,1}$ be a category whose objects are pairs (F, σ), where σ is an ordering of the n free boundary components of F, and a morphism from (F, σ) to (F', σ') is an isotopy class of orientation preserving diffeomorphisms from F to F' which preserve the ordering.

(B) Let $\mathcal{S}_{0,1,1}$ be a category with one object S^1 and the morphism set $\mathbb{Z}$.

REMARK 3.2. The reason why we take S^1 rather than a surface of type $F_{0,2}$ is that we need the identity element in the mapping class group operad. Note that the morphism set $\mathbb{Z}$ of $\mathcal{S}_{0,1,1}$ stands for $\Gamma_{0,2}$.

THEOREM 3.3. *$B\mathcal{S}_{g,n,1}$ has the same homotopy type as $B\Gamma_{g,n+1}$.*

Proof. It is easy to see that $B\mathcal{S}_{0,1,1} \simeq B\mathcal{C}_{\Gamma_{0,2}}^{\Gamma_{0,1,1}} = B\Gamma_{0,2}$. For two objects (F,σ) and (F',σ'), $Hom((F,\sigma),(F',\sigma'))$ is isomorphic to $\Gamma_{g,n+1}$ which we may regard as $Hom((F_0,\sigma),(F_0,\sigma))$ for a fixed surface F_0 and an ordering σ. Hence $\mathcal{S}_{g,n,1}$ is a groupoid that satisfies the assumption of Lemma 2.3. Thus $B\mathcal{S}_{g,n,1}$ has the same homotopy type as $B\Gamma_{g,n+1}$. $\square$

THEOREM 3.4. *Let $\mathcal{S}_n = \amalg_{g\geq 0} B\mathcal{S}_{g,n,1}$. Then $\mathcal{S} = \amalg_{n\geq 0}\mathcal{S}_n$ is an operad with the structure map induced by the attachings.*

This operad $\mathcal{S}$ is called the mapping class group operad. Theorem 3.3 says that $\mathcal{S} \simeq \amalg_{n\geq 0}(\amalg_{g\geq 0} B\Gamma_{g,n+1})$. Note also that $\amalg_{n\geq 0}(\amalg_{g\geq 0}\Gamma_{g,n+1})$ is a monoidal category whose product is induced by the pair of pants multiplication of two surfaces.

Proof of Theorem 3.4. The structure map on $\mathcal{S}$ is induced by the attachings

$$\mathcal{S}_{g,k,1} \int \mathcal{S}_{h,n,1} \longrightarrow \mathcal{S}_{g+kh,kn,1}$$

which is well-defined because each surface obtained by the attachings is an object in one of these categories. The single object S^1 of $\mathcal{S}_{0,1,1}$ plays the role of the identity element $1 \in \mathcal{S}_1$. Note that any attachings do not produce a surface of type $F_{0,1,1}$. The object D in $\mathcal{S}_{0,0,1}$ stands for $0 \in \mathcal{S}_0$. $\square$

REMARK 3.5. There are canonical maps of operads $\mathcal{B} \to \mathcal{RB} \to \mathcal{S}$ induced by inclusions. Note that $R\beta_n \cong \Gamma_{0,n,1}$ and $PR\beta_n \cong \Gamma_{0,n+1}$. We also note that $\mathcal{S}_{g,n,1}$ is equivalent to $\mathcal{C}_{\Gamma_{g,n+1}}^{\Gamma_{g,n,1}}$. The forgetful functor $\mathcal{S}_{g,n,1} \to \mathcal{C}_e^{\Sigma_n}$ induces a map of operad $\mathcal{S} \to \Gamma$, which implies that every Γ-space may be regarded as an $\mathcal{S}$-space via this map.

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