

## HARMONIC MAPPING

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ABSTRACT. In this paper, we obtain some coefficient bounds of harmonic, orientation-preserving, univalent mappings defined on  $\Delta = \{z : |z| > 1\}$ .

### 1. Introduction

A continuous function  $f = u + iv$  defined in a domain  $D \subseteq \mathbb{C}$  is harmonic in  $D$  if  $u$  and  $v$  are real harmonic in  $D$ . Let  $\Sigma$  be the set of all complex-valued, harmonic, orientation-preserving, univalent mappings

$$(1.1) \quad f(z) = h(z) + \overline{g(z)} + A \log|z|$$

of  $\Delta = \{z : |z| > 1\}$ , where

$$h(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in  $\Delta$  and  $A \in \mathbb{C}$ . Hengartner and Schober[3] used the representation (1.1) to obtain some coefficient estimates and distortion theorems. Some coefficient bounds for  $f \in \Sigma$  are also obtained by Jun[4] when  $f(\Delta) = \Delta$ .

In this article, we obtain some coefficient bounds of  $f \in \Sigma$  by using properties of the analytic function  $h - g$ .

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## 2. Some coefficient bounds

Let  $S$  be the class of functions  $H(z) = z + \sum_{n=0}^{\infty} c_n z^{-n}$  that are analytic and univalent for  $|z| > 1$ . The coefficient problems for this class appear to be difficult. One reason is that there can be no single extremal function for all coefficients. Sharp bounds for  $|c_n|$  are known only for  $1 \leq n \leq 3$ :

$$(2.1) \quad |c_1| \leq 1 \quad [2], \quad |c_2| \leq \frac{2}{3} \quad [5], \quad |c_3| \leq \frac{1}{2} + e^{-6} \quad [1].$$

Although information beyond the third coefficient is incomplete, we have some useful results proved by Schober and Williams[6].

**Theorem 2.1**[6]. *Let  $H \in S$ .*

- (1) *If  $c_1$  is real, then  $Re\{c_4 + 4c_1\} \leq 4$ .*
- (2) *If  $c_1$  and  $c_2$  are real, and  $c_1 \geq \frac{1}{3}$ , then  $Re\{c_5 + c_3 + c_1\} \leq 1$ .*
- (3) *If  $c_1$  and  $c_2$  are real, then  $Re\{c_5 + c_3 - 2c_1\} \leq 2$ .*
- (4) *If  $c_1$  is real, and  $c_2$  is imaginary, then  $Re\{5c_1 - c_5\} \leq 5$ .*

**Corollary 2.2.** *Let  $H \in S$ . Then we have*

$$(1) \quad Re\{c_4\} \leq 8 \text{ if } c_1 \text{ is real,}$$

$$(2) \quad Re\{c_5\} = \begin{cases} \leq \frac{7}{6} + e^{-6} & \text{if } c_1 \text{ and } c_2 \text{ are real, and } c_1 \geq \frac{1}{3} \\ \leq \frac{9}{2} + e^{-6} & \text{if } c_1 \text{ and } c_2 \text{ are real} \\ \geq -10 & \text{if } c_1 \text{ is real, and } c_2 \text{ is imaginary.} \end{cases}$$

*Proof.* By simple calculation, one can easily show this from (2.1) and Theorem 2.1.  $\square$

**Theorem 2.3.** *If  $f \in \Sigma$  and  $h - g$  is univalent, then*

$$|a_1 - b_1| \leq 1, \quad |a_2 - b_2| \leq \frac{2}{3}, \quad |a_3 - b_3| \leq \frac{1}{2} + e^{-6}.$$

*Proof.* Let  $H(z) = h(z) - g(z) = z + \sum_{n=1}^{\infty} c_n z^{-n}$ , where  $c_n = a_n - b_n$ . Then  $H(z)$  is univalent and analytic. Thus  $H(z) \in S$ . By using the sharp bounds for  $|c_n|$  in (2.1), we obtain our sharp estimates.  $\square$

**Theorem 2.4.** For each  $f \in \Sigma$  with univalent  $h - g$ , we have

$$(1) \operatorname{Re}\{a_4 - b_4\} \leq 8 \text{ if } a_1 - b_1 \text{ is real,}$$

$$(2) \operatorname{Re}\{a_5 - b_5\} = \begin{cases} \leq \frac{7}{6} + e^{-6} & \text{if } a_1 - b_1 \text{ and } a_2 - b_2 \text{ are real,} \\ & \text{and } a_1 - b_1 \geq \frac{1}{3} \\ \leq \frac{9}{2} + e^{-6} & \text{if } a_1 - b_1 \text{ and } a_2 - b_2 \text{ are real} \\ \geq -10 & \text{if } a_1 - b_1 \text{ is real,} \\ & \text{and } a_2 - b_2 \text{ is imaginary.} \end{cases}$$

*Proof.*  $H(z) = h(z) - g(z) \in S$ . Thus we have estimates by Corollary 2.2.  $\square$

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