CONVERGENCE OF PREFILTER BASE ON THE FUZZY SET

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Abstract. In this paper, we investigate the prefilter base on a fuzzy set and fuzzy net \( \varphi \) on the fuzzy topological space \((X, \delta)\). And we show that the prefilter base \( B(\varphi) \) determines by the fuzzy net \( \varphi \) converge to a fuzzy point \( p \) iff the fuzzy net \( \varphi \) converge to a fuzzy point \( p \). Also we prove that if the prefilter base \( B \) converge to a fuzzy point \( p \), then the \( B \) has the cluster point \( p \).

1. Introduction

The concept of a fuzzy set, which was introduced in [1], provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. Throughout this paper, the symbol \( I \) will denote the unit interval. Let \( X \) be a non-empty set. A fuzzy set in \( X \) is a function with domain \( X \) and value in \( I \), that is, an element of \( I^X \).

2. Preliminaries

Definition 1. A fuzzy point \( p \) in \( X \) in a fuzzy set with membership function:

\[
p(x) = \begin{cases} t_0, & \text{if } x = x_0 \\ 0, & \text{otherwise} \end{cases}
\]

where \( 0 < t_0 < 1 \). \( p \) is said to have support \( x_0 \) and value \( t_0 \), and is noted by \( p(x_0, t_0) \) or even \((x_0, t_0)\). We denote by \( B_F(X) \) the collection of all fuzzy points in \( X \).
DEFINITION 2. Let \( \{ \mu_i \mid i \in \Lambda \} \) be a fuzzy sets in \( X \). We define the following fuzzy sets:

(1) \( \wedge \{ \mu_i \mid i \in \Lambda \}(x) = \inf \{ \mu_i(x) \mid i \in \Lambda \} \) for each \( x \in X \).

(2) \( \vee \{ \mu_i \mid i \in \Lambda \}(x) = \sup \{ \mu_i(x) \mid i \in \Lambda \} \) for each \( x \in X \).

(3) \( c_t \in I^X \), by \( c_t(x) = t \) for each \( x \in X \) and \( t \in I \).

In 1968, C.L. Chang define a fuzzy topology on \( X \) as a subset \( \delta \subset I^X \) such that

(1) \( c_0, c_1 \in \delta \).

(2) If \( \mu_1, \mu_2 \in \delta \), then \( \mu_1 \wedge \mu_2 \in \delta \).

(3) If \( \{ \mu_\alpha \mid \alpha \in \Lambda \} \subset \delta \), then \( \vee \{ \mu_\alpha \mid \alpha \in \Lambda \} \in \delta \).

Several articles on the subject all involve this definition. Amongst these the most important ones are [3, 6]. It is concept of fuzzy topology that will be used throughout the sequel. Chang’s definition we will refer to as quasi fuzzy topology.

The fuzzy sets in \( \delta \) are called open fuzzy sets. A fuzzy set \( A \in I^X \) is called closed iff \( A^c \) is open.

DEFINITION 3. A prefilter \( \mathcal{F} \) on \( X \) is a nonempty collection of subsets of \( I^X \) with the properties:

(1) If \( F_1, F_2 \in \mathcal{F} \), then \( F_1 \wedge F_2 \in \mathcal{F} \).

(2) If \( F_1 \in \mathcal{F} \) and \( F_2 \geq F_1 \), then \( F_2 \in \mathcal{F} \).

(3) \( 0 \notin \mathcal{F} \).

DEFINITION 4. A collection \( \mathcal{B} \) of subsets of \( I^X \) is a prefilter base iff \( \mathcal{B} \neq \emptyset \) and

(1) If \( B_1, B_2 \in \mathcal{B} \) then \( B_3 \leq B_1 \wedge B_2 \) for some \( B_3 \in \mathcal{B} \);

(2) \( 0 \notin \mathcal{B} \).

The collection \( \mathcal{F} = \{ F \in I^X \mid \exists B \in \mathcal{B} \text{ s.t } F \geq B \} \) is prefilter. \( \mathcal{F} \) is said to be generated by \( \mathcal{B} \) and denoted \( < \mathcal{B} > \).

3. Converges and Cluster Point

A directed set \((D, \prec)\) is a set with partial order \( \prec \) such that for each pair \( a, b \) of elements of \( D \), there exists an element \( c \) of \( D \) having the property that \( a \prec c \) and \( b \prec c \).

Let \((D, \prec)\) be a directed set. The terminal set \( T_a \) determined by an \( a \in D \) is \( \{ b \in D \mid a \prec b \} \).
Let \((D, \prec)\) be a directed set. A fuzzy net in \(X\) is a map \(\varphi : D \to B_F(X)\).

If \(\varphi(b) = p_b(x_b, t_b)\), we also denote \(\varphi\) by \(\{\varphi(b) | b \in D\}\) or \(\{p_b | b \in D\}\).

From now on, \(x_b\) and \(t_b\) will be the support and the value of the fuzzy point \(p_b\). If \(a \in D\), the fuzzy set \(\varphi(T_a) = \bigvee\{\varphi(b) | a \prec b\}\) is called a \(F\)-tail of \(\varphi\).

**Definition 3.1.** Let \((X, \delta)\) be an f.t.s. Let \(\varphi\) be fuzzy nets on \(X\) and \(p(x, t) \in B_F(X)\). We say that:

1. \(\varphi\) converges to \(p\) (written \(\varphi \xrightarrow{} p\)), if for all \(N \in N_\delta \) \(\exists a \in D, \forall b \succ a \text{ s.t. } \varphi(b) \in N\).

2. \(p\) is a cluster point of \(\varphi\) (written \(\varphi \xrightarrow{c} p\)), if \(\forall N \in N_\delta, \forall a \in D, \exists b \succ a \text{ s.t. } \varphi(b) \in N\).

Let \(\varphi : D \to B_F(X)\) be a fuzzy net. Then the family \(B(\varphi) = \{\varphi(T_a) | a \in D\}\) is a prefilter base in \(X\). For, given \(\varphi(T_a)\) and \(\varphi(T_b)\), first find a \(c \in D\) such that \(a \prec c, b \prec c\), and then observe that \(T_c \leq T_a \land T_b\) because \(\prec\) is transitive. \(B(\varphi)\) is called the prefilter base determined by the fuzzy net \(\varphi\).

**Definition 3.2.** Let \((D, \prec)\) be a directed set and \(T_a\) be a terminal set. Then, for a fuzzy net \(\varphi : D \to B_F(X)\), we have:

1. \(\varphi\) converges to \(p\) (written \(\varphi \xrightarrow{} p\)), if for all \(N \in N_\delta\) \(\exists a \in D, \forall b \succ a \text{ s.t. } \varphi(b) \in N\).

2. \(p\) is a cluster point of \(\varphi\) (written \(\varphi \xrightarrow{c} p\)), if \(\forall N \in N_\delta, \forall a \in D, \exists b \succ a \text{ s.t. } \varphi(b) \in N\).

**Definition 3.3.**

1. A prefilter \(\mathcal{F}\) is said to converge to the fuzzy point \(p\) (written \(\mathcal{F} \xrightarrow{} p\)) iff \(N_\delta \subset \mathcal{F}\), that is, \(\mathcal{F}\) is finer than the nhood prefilter at \(p\).

2. We say \(\mathcal{F}\) has \(p\) as a cluster point (written \(\mathcal{F} \xrightarrow{c} p\)) iff \(\forall N \in N_\delta, \forall T_a, \varphi(T_a) \land N \neq 0\).

We can express these notions in terms of prefilter base as follows:

1. A prefilter base converges to a fuzzy point \(p\) (\(\mathcal{B} \xrightarrow{} p\)) iff each \(N \in N_\delta\) contains some \(B \in \mathcal{B}\).

2. A prefilter base has \(p\) as a cluster point (\(\mathcal{B} \xrightarrow{c} p\)) iff each \(N \in N_\delta\) meets each \(B \in \mathcal{B}\).

These definitions are still valid if we use nhood bases at \(p, B_\delta^p\), instead of nhood systems as \(p, N_\delta^p\). Clearly, if \(\mathcal{F} \xrightarrow{} p\), then \(\mathcal{F} \xrightarrow{c} p\).
Lemma 3.4. Let $B(\varphi)$ be the prefilter base determined by the fuzzy net $\varphi : D \rightarrow B_F(X)$: Then

1. $\varphi \rightarrow p \text{ iff } B(\varphi) \rightarrow p$.
2. $\varphi \propto p \text{ iff } B(\varphi) \propto p$.

Theorem 3.5. Let $(X, \delta)$ be a f.t.s., and $F$ a prefilter on $X$ and $B$ a prefilter base of $F$. Then $B$ converges to a fuzzy point $p$ iff $F$ converges to the fuzzy point $p$.

Proof. Let $B$ converges to a fuzzy point $p$. Since there exists $B_\alpha \in B$ such that $B_\alpha \leq N$ for all $N$ in $\mathbb{N}_p$. Then $N \in F$ and $N_\delta \subset F$. Hence $F \rightarrow p$. Conversely, if $F \rightarrow p$, then $N_\delta \subset F$ and since $B$ is prefilter base $F$. There exists $B_\alpha \in B$ such that $B_\alpha \leq N \in F$ for all $N \in N_\delta \subset F$. Hence $B \rightarrow p$. □

Definition 3.6. Let $U = \{A_\alpha | \alpha \in \Lambda\}$ and $B = \{B_\beta | \beta \in \Gamma\}$ be two prefilter bases on $X$. $B$ is subordinate to $U$, written $B \vdash U$, if there exist $B_\beta \in B$ such that $B_\beta \leq A_\alpha$ for all $A_\alpha \in U$.

Theorem 3.7. Let $U = \{A_\alpha | \alpha \in \Lambda\}$ and $B = \{B_\beta | \beta \in \Gamma\}$ be two prefilter bases on $X$.

1. If $U \subset B$ then $B \vdash U$.
2. If $B \vdash U$, then each member of $B$ meets every member of $U$.

Proof. (1) is obvious. (2). Assume there exist $B_\beta, A_\alpha$ such that $A_\alpha \wedge B_\beta = 0$; since $B \vdash U$, for this $A_\alpha$ we can find a $B_\gamma \leq A_\alpha$, and then $B_\gamma \wedge B_\beta = 0$ contradicts that $B$ is a prefilter base. □

Theorem 3.8. An f.t.s. $(X, \delta)$ is Hausdorff iff each convergent prefilter base in $X$ converges to exactly one fuzzy point.

Proof. Assume that $X$ is fuzzy Hausdorff and $B$ is a prefilter base and $B \rightarrow p$. For any pair of distinct fuzzy points $p, q$ in $X$. Then there exists fuzzy open set $\mu, \nu$ in $I^X$ such that $p \in \mu$, $q \in \nu$ and $\mu \wedge \nu = 0$. Since by hypothesis there is some $B_1 \leq \mu$ and and since any two $B_1, B_2$ have nonempty intersection, there can be no $B_2 \leq \nu$, thus, $B$ cannot converge to $q \neq p$.

Conversely, assume that $X$ is not Hausdorff. Then there must exist $p, q$ such that $N \wedge M \neq 0$ for all $N$ in $N_\delta^p$ and $M$ in $N_\delta^q$. 
Theorem 3.9. Let \( \mathcal{U} = \{ A_\alpha | \alpha \in \Lambda \} \) and \( \mathcal{B} = \{ B_\beta | \beta \in \Gamma \} \) be two prefilter bases on \( X \).

(1) \( \mathcal{U} \not\rightarrow p \Rightarrow \mathcal{U} \not\propto p \) and, in Hausdorff spaces, at no point other than \( p \).

(2) Let \( \mathcal{B} \vdash \mathcal{U} \). Then;

(a) \( \mathcal{U} \not\rightarrow p \Rightarrow \mathcal{B} \not\rightarrow p \)

(b) \( \mathcal{B} \not\propto p \Rightarrow \mathcal{U} \not\propto p \)

Proof. (1). Given \( N \in N^\delta_p \), there is some \( A_\alpha \in \mathcal{U} \) such that \( A_\alpha \leq N \); since each \( A_\beta \) must intersect \( A_\alpha \), it follows that \( A_\beta \land N \neq 0 \) for all \( A_\beta \), so \( \mathcal{U} \not\propto p \).

Now let \( X \) be fuzzy Hausdorff and let \( p \neq q \); choosing disjoint fuzzy nbds \( N \in N^\delta_p, M \in N^\delta_q \), there must be some \( A_\alpha \in \mathcal{U} \) contained in \( N \); then \( A_\alpha \land M = 0 \), and so \( \mathcal{U} \) cannot cluster point at \( q \).

(2a). There is some \( A_\alpha \in \mathcal{U} \) such that \( A_\alpha \leq N \) for all \( N \in N^\delta_p \); since \( \mathcal{B} \vdash \mathcal{U} \), there is a \( B_\beta \leq A_\alpha \), so \( \mathcal{B} \rightarrow p \) also.

(2b). Given \( N \in N^\delta_p \) and \( A_\alpha \), there is some \( B_\beta \leq A_\alpha \), and since \( B_\beta \land N \neq 0 \) for all \( B_\beta \), we can find \( A_\alpha \land N \neq 0 \) for all \( A_\alpha \), which proves \( \mathcal{U} \not\propto p \). □

As a immediately, we have the follows.

Corollary 3.10.

(1) \( \mathcal{U} \not\rightarrow p \) iff \( \forall \mathcal{B} \vdash \mathcal{U}, \exists \mathcal{C} \vdash \mathcal{B} \text{ s.t } \mathcal{C} \rightarrow p \)

(2) \( \mathcal{U} \not\propto p \) iff \( \exists \mathcal{B} \vdash \mathcal{U} \text{ s.t } \mathcal{B} \rightarrow p \)

Let \( \mathcal{B} \) be a prefilter base on \( X \). We say that a subset \( Y \) of \( X \) contains \( \mathcal{B} \) if every member of \( \mathcal{B} \) is an element of fuzzy set with support \( Y \).

Definition 3.11. Let \( \mathcal{M} \) be a prefilter base in \( X \) is called fuzzy maximal if it has no property subordinated prefilter base, that is, if for all \( \mathcal{U}, \mathcal{U} \vdash \mathcal{M} \Rightarrow \mathcal{M} \vdash \mathcal{U} \).

Theorem 3.12. A prefilter base \( \mathcal{M} \) is fuzzy maximal iff for each \( Y \subset X \), either \( Y \) or \( Y^c \) contains a member of \( \mathcal{M} \).
Proof. Assume $\mathcal{M} = \{M_\beta | \beta \in \Gamma\}$ is a fuzzy maximal prefilter base. Let $Y \subset X$ and $A$ be a fuzzy set with support $Y$ and $B$ is a fuzzy set with support $Y^c$. Then we cannot have an $Y$ contains $M_\beta$ and $Y^c$ contains $M_\gamma$, since $M_\beta \land M_\gamma = 0$. Assume now that $Y$ not contains $M_\beta$ for all $M_\beta$. Then $A \land M_\beta \neq 0$ and also then all $M_\beta \land B \neq 0$, so $\mathcal{U} = \{M_\beta \land B | M_\beta \in \mathcal{M}\}$ is a prefilter base. Since $\mathcal{U} \vdash \mathcal{M}$, also $\mathcal{M} \vdash \mathcal{U}$. Hence we have $M_\gamma \leq M_\beta \land B \leq B$. Therefore $M_\gamma \leq B$, that is $Y^c$ contains $M_\gamma$.

Conversely, assume that for each $Y \subset X$, $Y$ or $Y^c$ contains a member of $\mathcal{M}$ and that $\mathcal{U} \vdash \mathcal{M}$. Given any $A \in \mathcal{U}$, the condition assures that either there is an $M_\beta \leq A$ or $M_\beta \leq B$ the latter possibility is excluded, since the assumption $\mathcal{U} \vdash \mathcal{M}$ implies [Theorem 3.7 (2)] that all $M_\beta \land A \neq 0$. Thus $\mathcal{M} \vdash \mathcal{U}$ and $\mathcal{M}$ is fuzzy maximal. □

Corollary 3.13. Let $\mathcal{M}$ be a fuzzy maximal prefilter base in $X$. Then $\mathcal{M} \propto p$ iff $\mathcal{M} \rightarrow p$.

Proof. Only the implication $(\mathcal{M} \propto p) \Rightarrow (\mathcal{M} \rightarrow p)$ need be proved. Given $N \in N_\delta$, there is an $M_\alpha \leq N$ or an $M_\alpha \leq N^c$; since $\mathcal{M} \propto p$, so that $M_\alpha \land N \neq 0$ for each $M_\alpha$, the latter possibility is excluded, and therefore $\mathcal{M} \rightarrow p$. □

References


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