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ON BROWDER'S THEOREM

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Abstract. In this paper we give several necessary and sufficient conditions for an operator on the Hilbert space to obey Browder's theorem. And it is shown that if S has totally finite ascent and $T \prec S$ then $f(T)$ obeys Browder's theorem for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$.

1. Introduction

Throughout this note let $B(H)$ and $K(H)$ denote respectively the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional Hilbert space H. If $T \in B(H)$ write $N(T)$ and $R(T)$ for the null space and range of T; $\sigma(T) = \dim N(T)$; $\beta(T) = \dim N(T^*)$; $\sigma(T)$ for the spectrum of $T; \pi_0(T)$ for the set of eigenvalues of $T; \pi_{0f}(T)$ for the eigenvalues of finite multiplicity; $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity; $p_{00}(T) = \sigma(T)\backslash\sigma_b(T)$ for *the Riesz points* of T. An operator $T \in B(H)$ is called *Fredholm* if it has closed range with finite dimensional null space and its range of finite co-dimension. The *index* of a Fredholm operator is given by

$$i(T) = \sigma(T) = \alpha(T) - \beta(T).$$

An operator $T \in B(H)$ is called *Weyl* if it is Fredholm of index zero. An operator $T \in B(H)$ is called Browder if it is Fredholm "of finite ascent and descent": equivalently ([9], Theorem 7.9.3]) if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum, the Weyl spectrum $\sigma_e(T)$ and the Browder spectrum $w(T)$ and

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the Browder spectrum of $\sigma_b(T)$ of $T \in B(H)$ are denoted by ([8],[9])

$$\sigma_e(T) = \{\lambda \in \ : T - \lambda I \text{ is not Fredholm } \};$$

$$w(T) = \{\lambda \in \ : T - \lambda I \text{ is not Weyl } \};$$

$$\sigma_e bT) = \{\lambda \in \ : T - \lambda I \text{ is not Browder } \} :$$

evidently

$$\sigma_b(T) \subseteq w(T) \subseteq \sigma_b(T) = \sigma(T) \cup acc\ \sigma(T),$$

where we write $\operatorname{acc} K$ for the accumulation points of $K \subseteq \mathbb{C}$.

We say that *Weyl's theorem holds for* $T \in B(H)$ if

$$\sigma(T) \setminus w(T) = \pi_{00}(T), \tag{1.1}$$

and that *Browder's theorem holds for* $T \in B(H)$ if

$$\sigma(T) \setminus w(T) = p_{00}(T). \tag{1.2}$$

An opeator $T \in B(H)$ is a G_m-operator $(m \geq 1)$ if there exists a constant M such that

$$\| (T - \lambda I)^{-1} \| \leq \frac{M}{(d(\lambda, \sigma(T))))^m} \text{ for every } \lambda \notin \sigma(T).$$

The condition N_λ is said to be satisfied at a particular λ if

$$N(T - \lambda I) \cap N([((T - \lambda I)^*)^n]$$

is nontrivial for some positive integer n, which may depend on λ.

An operator $T \in B(H)$ is said to be dominant if for every $\lambda \in \mathbb{C}$ there exists a constant M_λ such that

$$(T - \lambda I)(T - \lambda I)^* \leq M_\lambda (T - \lambda I)^*(T - \lambda I)$$

and an operator $T \in B(H)$ is said to be paranormal if

$$\| Tx \|^2 \leq \| T^2 x \| \text{ for all } x \in H.$$

In particular, T is called totally paranormal if $T - \lambda I$ is paranormal for every $\lambda \in \mathbb{C}$. $X \in B(H)$ is called a quasiaffinity if it has trivial kernel and dense range. $S \in B(H)$ is said to be a quasiaffine transform of $T \in B(H)$ (notation:$S \prec T$) if there is a quasiaffinity $X \in B(H)$ such that $XS = TX$. If both $S \prec T$ and $T \prec S$, then we say that S and T are quasisimilar. An operator $T \in B(H)$ has *totally finite ascent* if $T - \lambda I$ has finite ascent for each $\lambda \in C$. It is known that if $T \in B(H)$ then we have :

$$\text{Weyl's theorem} \Rightarrow \text{Browder's theorem}.$$

2. Main Results

THEOREM 2.1. *Let $T \in B(H)$. Then the following statements are equivalent:*

(i) *T obeys Browder's theorem;*
(ii) *$\sigma(T)\backslash w(T) \subset iso\ \sigma(T)$;*
(iii) *$\gamma_T(\lambda)$ is discontinuous for each $\lambda \in \sigma(T)\backslash w(T)$, where $\gamma_T(\cdot)$ denotes the reduced minimum modulus;*
(iv) *Every $\lambda \in \alpha(T-\lambda_1 I)$ satisfies the condition N_λ;*
(v) *$T-\lambda I$ has finite ascent for each $\lambda \in \sigma(T)\backslash w(T)$.*

Proof. (i) $\Leftrightarrow$ (ii) : If T obeys Browder's theorem then

$$\lambda \in \sigma(T)\backslash w(T) = p_{00}(T) \subset iso\ \sigma(T).$$

Conversely, suppose $\lambda \in \sigma(T)\backslash w(T)$. Then $T-\lambda I$ is Weyl. But $\lambda \in iso\ \sigma(T)$; hence by the punctured neighborhood theorem $\lambda \in \sigma_b(T)$. Therefore T obeys Browder's theorem.

(ii) $\Leftrightarrow$ (iii) : If T obeys Browder's theorem then it follows from [**6**, Lemma 5.52] that $\gamma_T(\lambda)$ is discontinuous for each $\lambda \in \sigma(T)\backslash w(T)$. Conversely, suppose $\gamma_T(\lambda)$ is discontinuous for each $\lambda \in \sigma(T)\backslash w(T)$. Let $\lambda_0 \in \sigma(T)\backslash w(T)$. Then $T-\lambda_0 I$ is Weyl and $\alpha(T-\lambda_0 I) > 0$. Therefore $\gamma_T(\lambda) > 0$ for all λ near λ_0, and so by [**6**, Cor 5.74] $\alpha(T-\lambda I) < \alpha(T-\lambda_0 I)$; for otherwise $\gamma_T(\lambda)$ would be continuous at λ_0. Since all nearby walues λ are also in $\sigma(T)\backslash w(T)$, the discontinuity of $\gamma_T(\lambda)$ requires that $\alpha(T-\lambda I) = 0$ in $\sigma(T)\backslash w(T)$. Therefor λ_0 is an isolated point of $\sigma(T)$.

(i) $\Leftrightarrow$ (iv) : The forward implication follows from [**7**, Theorem 1]. Conversely, suppose $\lambda_0 \in \sigma(T)\backslash w(T)$. Then $T-\lambda_0 I$ is Weyl and $\alpha(T-\lambda I > 0)$. Since every $\lambda \in \sigma(T)\backslash w(T)$ satisfies the condition N_λ, by the punctured neighborhood theorem there exists a neighborhood $N(\lambda_0 : p)$ for some $p > 0$ such that $\alpha(T-\lambda I)$ is constant (say n_0) on $N(\lambda_0 : p\backslash\{\lambda_0\})$ and $0 \leq \alpha(T-\lambda I) < \alpha(T-\lambda_0 I)$. We now claim that $n_0 = 0$. Assume to the contrary that $n \neq 0$. Also by the punctured neighborhood theorem there exists a neighborhood $N(\lambda_0 : q)$ for some $q > 0$ such that $\lambda_1 \in N(\lambda_0 : q)\backslash\{\lambda_0\}$ implies $\alpha(T-\lambda_1 I) > 0$ and $T-\lambda_1 I$ is Weyl. Thus we have $\lambda_1 \in \sigma(T)\backslash w(T)$. Now by the same reason as for λ_0, there exists a neighborhood $N(\lambda_1 : \gamma)$ for some $\gamma > 0$ such that $\alpha(T-\mu)$ is constant (say n_1) and $0 \leq \alpha(T-\mu) < \alpha(T-\lambda_1 I)$. Thus

$$\lambda \in [N(\lambda_0 : q) \cap N(\lambda_1 : \gamma)]\backslash\{\lambda_0, \lambda_1\} \Rightarrow \alpha(T-\lambda I) = n_1 < n_0,$$

a contradiction, Therefore n_0 and hence λ is an isolated point of $\sigma(T)$. Hence it follows from (ii) that Browder's theorem holds for T.

(i) $\Leftrightarrow$ (v) : if T obeys Browder's theorem then $\sigma(T)\backslash w(T) = p_{00}(T)$. Therefore $T - \lambda I$ has finite ascent for each $\lambda \in \sigma(T)\backslash w(T)$. Conversely, suppose $T - \lambda I$ has finite ascent for each $\lambda \in \sigma(T)\backslash w(T)$. Then by the Index Product Theorem,

$$\alpha((T - \lambda I)^n) - \beta((T - \lambda I)^n) = i((T - \lambda I)^n) = n \cdot i(T - \lambda I) = 0.$$

Thus if $\alpha((T - \lambda I)^n)$ is a constant then so is $\beta((T - \lambda I)^n)$. Therefore $T - \lambda I$ is Browder. Thus T obeys Browder's theorem. $\square$

We can't expect that Weyl's theorem holds for operators having totally finite ascent. Consider the following example: let $T \in B(l_2)$ be defined by

$$T(x_1, x_2, x_3 \cdots) = (0, x_1, \frac{1}{2}x_2 \frac{1}{3}x_3, \cdots).$$

Then T is a dominant operator, and so T has totally finite ascent. But $\sigma(T) = w(T) = \{0\}$ and $\pi_{00}(T) = \phi$; Hence Weyl's theorem doesn't hold for T. However, Browder's theorem performs better:

Corollary 2.2. *Suppose $S \in B(H)$ has totally finite ascent and $T \in B(H)$ satisfies $T \prec S$. Then $f(T)$ obeys Browder's theorem for every $f \in H(\sigma(T))$. In particular if S is a dominant operator and $T \prec S$ then Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set for all analytic functions on an open neighborhood of $\sigma(T)$.*

Proof. Since $T \prec S$, there exists a quasiaffinity $X \in B(H)$ such that $XT = SX$. But S has totally finite ascent; hence for each λ there exists a natural number n_λ such that $N((S-\lambda I)^{n_\lambda}) = N((S-\lambda I)^{n_\lambda+1})$. We claim that $N((T-\lambda I)^{n_\lambda}) = N((T-\lambda I)^{n_\lambda+1})$. Let $x \in N((T-\lambda I)^{n_\lambda+1})$. Then $N(T - \lambda I)^{n_\lambda+1}x = 0$, and so $(S - \lambda I)^{n_\lambda+1}Xx = X(T - \lambda I)^{n_\lambda+1}x = 0$. Then $(T-\lambda I)^{n_\lambda+1}x = 0$, and so $(S-\lambda I)^{n_\lambda+1}Xx = X(T-\lambda I)^{n_\lambda+1}x = 0$. Therefore, $Xx \in N((S - \lambda I)^{n_\lambda+1}) = N((S - \lambda I)^{n_\lambda})$, and so. $(S - \lambda I)^{n_\lambda}Xx = 0$. Since $X(T - \lambda I)^{n_\lambda}x = 0$ and X is a quasiaffnity $x \in N(T - \lambda I)^{n_\lambda}$. Since T has totally finite ascent, it follows from Theorem 2.1 that . $w(T) = \sigma_b(T)$. Let $f \in H(\sigma(T))$. We shall show that $w(f(T)) = \sigma_b(f(T))$. Since $w(f(T)) \subset f(w(T))$ for every $f \in H(\sigma(T))$ with no other restriction on ([5, Theorem 2]), it suffices to show that

$f(w(T)) \subset w(f(T))$. Suppose $\lambda \notin w(f(T))$. Then $f(T) - \lambda I$ is Weyl and

$$f(T) - \lambda I = c(T - \alpha_1 I)(T - \alpha_2 I) \cdots (T - \alpha_n I)g(T), \tag{2.3}$$

where $c, \alpha_1, \alpha_1, \cdots, \alpha_1 \in \mathbb{C}$ and $g(T)$ is invertible. Since the operators in the right side of (2.3) commute, $T - \alpha_i$ is Fredholm. Now we show that $i(T - \alpha_i) \leq 0$. Observe that if $A \in B(H)$ is Fredholm of finite ascent then $i(A) \leq 0$: indeed, either if A has finite descent then A is Browder and hence $i(A) = 0$, or if A does not have finite descent then

$$n \cdot i(A) = \alpha(A^n) - \beta(A^n) \to -\infty \text{ as } N \to -\infty,$$

which implies that $i(A) < 0$. Therefore $\lambda \notin w(f(T))$, and hence $f(w(T)) = w(f(T))$. Hence $\sigma_b(f(T)) = f(\sigma_b(T)) = f(w(T)) = w(f(T))$, and so Browder's theorem holds for $f(T)$. If S is a dominant operator, then $N(S - \lambda I) \subset N(S - \overline{\lambda} I)$ for all $\lambda \in \mathbb{C}$. Therefore S has totally finite ascent, and hence the conclusion is evident from the previous assertion. □

Corollary 2.3. *Let $T \in B(H)$ be a G_m-operator. If T has totally finite ascent then $f(T)$ obeys Weyl's theorem for every $f \in H(\sigma(T))$.*

Proof. Since T has totally finite ascent, it follows from Theorem 2.1 that T obeys Browder's theorem. But T is a G_m operator, it follows from [10, Theorem 14] that T obeys Weyl's theorem. Let $f \in H(\sigma(T))$. Then by Corollary 2.2 $f(w(T)) = w(f(T))$ Remembering([13, Lemma]) that if T is isoloid then

$$f(\sigma(T) \backslash \pi_{00}(T)) = \sigma(f(T)) \backslash \pi_{00}(f(T))$$

for every $f \in H(\sigma(T))$. Hence

$$\sigma(f(T)) \backslash \pi_{00}(f(T)) = f(\sigma(T) \backslash \pi_{00}(T)) = f(w(T)) = w(f(T)),$$

which implies that Weyl's theorem holds for $f(T)$. □

Recall that if $T \in B(H)$ and F is a closed subset of then we define a *spectral subspace* as follows :

$$H_T(F) =$$
$$\{x \in H \mid (T - \lambda I)f(\lambda) = x \text{ has an analytic solution } f : C \backslash F \to H\}$$

Theorem 2.4. *Let $T \in B(H)$. If $H_T(\{\lambda\}) = N(T - \lambda I)$ for every $\lambda \in \pi_{0f}(T)$, then T obeys Weyl's theorem.*

Proof. Let $\lambda \in \sigma(T)\backslash w(T)$. Then $\lambda \in \pi_{0f}(T)$, and so $H_T(\{\lambda\}) = N(T-\lambda I)$. Since $H_T(\{\lambda\})$ is invariant under T, T can be represented as the following 2×2 operator matrix with respect to the decomposition $H_T(\{\lambda\}) \bigoplus H_T(\{\lambda\})^{\perp}$:

$$T = \begin{pmatrix} \lambda & T_1 \\ 0 & T_2 \end{pmatrix}.$$

Since $H_T(\{\lambda\})$ is finite dimensional, $T_2 - \lambda I$ is invertible $\lambda \in iso\ \sigma(T)$, and hence $\lambda_{00}(T)$. Conversely, let $\lambda_{00}(T)$. Then using the spectral projection,

$$P = \frac{1}{2\pi i}\int_{\partial D}(T-\lambda I)^{-1}d\lambda,$$

where D is an open disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T)\backslash\{\lambda\}.$$

Since $P(H) = \{x \in H : \lim \|T-\lambda I)^n x\|^{\frac{1}{n}} = 0\} = H_T(\{\lambda\})$ and $H_T(\{\lambda\})$ is finite dimensional, $w(T) = w(T_2)$. But $T - \lambda I$ is invertible; hence $T - \lambda I$ is Weyl. Therefore $\lambda \in \sigma(T)\backslash w(T)$. □

Corollary 2.5. *If $T \in B(H)$ is a totally paranormal operator then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. If T is totally paranormal, then it follows from [11, Corollary 4.8] that $H_T(\{\lambda\}) = N(T-\lambda I)$ for every $\lambda \in \mathbb{C}$. Therefore by Theorem 2.4 Weyl's theorem holds for T. But T has totally finite ascent and T is an isoloid; it follows follows from the proof of Corollary 2.3 that f(T) obeys Weyl's theorem. □

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