SOME PROPERTIES OF MV-ALGEBRAS

JUNG MI KO AND YONG CHAN KIM

Abstract. In this paper, we obtain an algebraic structure which is equivalent to an MV-algebra. Moreover, we show that t-norm and t-conorm can be obtained from MV-algebras.

1. Introduction

Ward and Dilworth [9] introduced residuated lattices as the foundation of the algebraic structures of fuzzy logics. Hájek [1] introduced a BL-algebra which is a general tool of a fuzzy logic. Recently, Hohle [2,3] extended the fuzzy set $f : X \rightarrow L$ where $L$ is an MV-algebra in stead of an unit interval $I$ or a lattice $L$.

In this paper, we obtain an algebraic structure which is equivalent to an MV-algebra. Moreover, we show that t-norm and t-conorm can be obtained from MV-algebras.

2. Preliminaries

Definition 2.1 ([3,8]). A lattice $(L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a residuated lattice if it satisfies the following conditions: for each $x, y, z \in L$,

(R1) $(L, \odot, 1)$ is a commutative monoid,
(R2) if $x \leq y$, then $x \odot z \leq y \odot z$ ( $\odot$ is an isotone operation),
(R3) (Galois correspondence): $(x \odot y) \leq z$ iff $x \leq (y \rightarrow z)$.

In a residuated lattice $L$, $x^* = (x \rightarrow 0)$ is called complement of $x \in L$.

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DEFINITION 2.2 ([3,8]). A residuated lattice \((L, \leq, \land, \lor, \circ, \rightarrow, 0, 1)\) is called a BL-algebra if it satisfies the following conditions: for each \(x, y \in L\),
\[(B1) \ x \land y = x \circ (x \rightarrow y), \]
\[(B2) \ x \lor y = [(x \rightarrow y) \rightarrow y] \land [(y \rightarrow x) \rightarrow x], \]
\[(B3) \ (x \rightarrow y) \lor (y \rightarrow x) = 1. \]
A BL-algebra \(L\) is called an MV-algebra if \(x = x^{**}\) for each \(x \in L\).

LEMMA 2.3 ([3,8]). Let \(L\) be an MV-algebra. For \(x, y, z \in L\), we have the following properties:
\[(1) \ x = (1 \rightarrow x), \]
\[(2) \ 1 = (x \rightarrow x), \]
\[(3) \ x \leq y \iff 1 = (x \rightarrow y), \]
\[(4) \ x = y \iff 1 = (x \rightarrow y) = (y \rightarrow x), \]
\[(5) \text{if } y \leq z, \ (x \rightarrow y) \leq (x \rightarrow z), \]
\[(6) \ (x \circ y) \rightarrow z = x \rightarrow (y \rightarrow z), \]
\[(7) \ x \circ y = (x \rightarrow y^*)^*, \]
\[(8) \ x \leq y \iff x^* \geq y^*, \]
\[(9) \ x \rightarrow y = y^* \rightarrow x^*. \]

DEFINITION 2.4 ([10]). A binary operation \(\otimes : [0, 1] \times [0, 1] \rightarrow [0, 1]\) is called a t-norm if it satisfies the following conditions: for each \(x, y, z \in L\),
\[(1) \ x \otimes y = y \otimes x, \]
\[(2) \ x \otimes (y \otimes z) = (x \otimes y) \otimes z, \]
\[(3) \ x \otimes 1 = x, \]
\[(4) \text{if } x \leq y, \ x \otimes z \leq y \otimes z. \]
We define the t-conorm as a dual sense of t-norm.

DEFINITION 2.5 ([10]). A binary operation \(\uplus : [0, 1] \times [0, 1] \rightarrow [0, 1]\) is called a t-conorm if it satisfies the following conditions: for each \(x, y, z \in L\),
\[(1) \ x \uplus y = y \uplus x, \]
\[(2) \ x \uplus (y \uplus z) = (x \uplus y) \uplus z, \]
\[(3) \ x \uplus 0 = x, \]
\[(4) \text{if } x \leq y, \ x \uplus z \leq y \uplus z. \]
3. Some properties of MV-algebras

**Theorem 3.1.** Let \((L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)\) be an MV-algebra. Define \(x \oplus y = x^* \rightarrow y\). For each \(x, y, z \in L\), we have the following properties.

1. \(x^{**} = x, 1^* = 0\).
2. \((x \oplus y)^* = x^* \odot y^*, (x \odot y)^* = x^* \oplus y^*\).
3. \(x \odot y = y \odot x, x \odot y = y \odot x\).
4. \(x \oplus (y \odot z) = (x \oplus y) \odot z, x \odot (y \odot z) = (x \odot y) \odot z\).
5. \(x \odot x^* = 1, x \odot x^* = 0\).
6. \(x \odot 0 = x, x \odot 1 = x\).
7. \(x \odot (x^* \oplus y) = y \odot (y^* \odot x), x \odot (x^* \odot y) = y \odot (y^* \odot x)\).
8. \(x \odot [y \odot (y^* \odot z)] = (x \odot y) \odot [(x^* \odot y^*) \odot (x \odot z)]\).
9. If \(y \leq z\), then \(x \odot y \leq x \odot z\).

**Proof.**

1. Since \(1 \rightarrow 0 = 0\) from Lemma 2.3(1), \(1^* = 0\).
2. Put \(z = 0\) from Lemma 2.3(6). Then \((x \odot y)^* = x \rightarrow y^* = x^* \oplus y^*\).

Furthermore, \((x \odot y)^* = (x^* \rightarrow y)^* = x^* \odot y^*\) from Lemma 2.3(7).

3-4. Since \((L, \odot)\) is a commutative monoid, that is, \(x \odot y = y \odot x\) and \(x \odot (y \odot z) = (x \odot y) \odot z\), by (2), \(x \odot y = y \odot x\) and \(x \odot (y \odot z) = (x \odot y) \odot z\).

5. By (B1), \(0 = x \wedge 0 = x \odot (x \rightarrow 0) = x \odot x^*\). It implies \(x \odot x^* = 1\).

6. Put \(y = 1\) from Lemma 2.3(7). Then \(x \odot 1 = (x \rightarrow 1^*)^* = x^{**} = x\). Moreover, by (2), \(x \odot 0 = x\).

7. By (B1), \(x \odot (x^* \oplus y) = x \odot (x \rightarrow y) = x \wedge y = y \wedge x = y \odot (y^* \odot x)\).

By (2), trivially, \(x \odot (x^* \odot y) = y \odot (y^* \odot x)\).

8. If \(y \leq z\), by Lemma 2.3(8), \(y^* \geq z^*\). By (R2), \(x^* \odot y^* \geq x^* \odot z^*\).

By Lemma 2.3(7-8), \((x^* \rightarrow y)^* \geq (x^* \rightarrow z)^*\) implies \(x^* \rightarrow y \leq x^* \rightarrow z\).

Since \(y \wedge z \leq y, z\), we have

\[
x^* \rightarrow (y \wedge z) \leq (x^* \rightarrow y) \wedge (x^* \rightarrow z).
\]

Since \((x^* \rightarrow y) \wedge (x^* \rightarrow z) \leq (x^* \rightarrow y), (x^* \rightarrow z), \) by (R3), \(x^* \odot ((x^* \rightarrow y) \wedge (x^* \rightarrow z)) \leq y, z\). Then \(x^* \odot ((x^* \rightarrow y) \wedge (x^* \rightarrow z)) \leq y \wedge z\). It implies \((x^* \rightarrow y) \wedge (x^* \rightarrow z) \leq x^* \rightarrow (y \wedge z)\). So,

\[
(x^* \rightarrow y) \wedge (x^* \rightarrow z) = x^* \rightarrow (y \wedge z).
\]

Thus,

\[
(x \oplus y) \wedge (x \oplus z) = x \oplus (y \wedge z).
\]
Since $x \wedge y = x \odot (x \rightarrow y)$ from (B1), we obtain

$$x \oplus [y \odot (y^* \oplus z)] = (x \oplus y) \odot [(x^* \odot y^*) \oplus (x \odot z)].$$

(9) Let $y \leq z$. By (2) and (8),

$$x \oplus y = x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z).$$

Hence, $x \oplus y \leq x \oplus z$.  \[\square\]

We can obtain the following corollary from Theorem 3.1.

**Corollary 3.2.** If $([0, 1], \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is an MV-algebra, then $([0, 1], \odot)$ is a t-norm and $([0, 1], \oplus)$ is a t-conorm.

**Theorem 3.3.** Let $(L, \odot, \oplus, \star, 0, 1)$ be an algebraic structure which satisfies (1)-(8) in Theorem 3.1. Define

$$x \leq y \iff x^* \oplus y = 1$$

$$x \rightarrow y = x^* \odot y$$

Then $(L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is an MV-algebra.

**Proof.** (1) $(L, \leq)$ is a partially ordered set.

(reflexive) Since $x^* \oplus x = 1$, $x \leq x$.

(transitive) If $x \leq y$ and $y \leq z$, then $x^* \oplus y = 1$ and $y^* \oplus z = 1$. Since $0 \leq 1$, $1 = 0^* \oplus 1 = 1 \oplus 1$. It implies $(x^* \odot y) \odot (y^* \odot z) = x^* \odot z = 1$. Thus $x \leq z$.

(anti-symmetric) If $x \leq y$ and $y \leq x$, then $x^* \oplus y = 1$ and $y^* \oplus x = 1$.

By Theorem 3.1(7), we have

$$x = x \odot 1 = x \odot (x^* \odot y) = y \odot (y^* \odot x) = y \odot 1 = y.$$

(2) We will show that $x \wedge y = x \odot (x^* \odot y)$.

Since $[x \odot (x^* \odot y)]^* \oplus x = [x^* \odot (x \odot y^*)] \oplus x = (x^* \oplus x) \odot (x \odot y^*) = 1 \oplus (x \odot y^*) = 1$ because $0 \leq (x \odot y^*)$, we have $x \odot (x^* \odot y) \leq x$. Since $[y \odot (y^* \odot x)]^* \odot y = [y^* \odot (y \odot x^*)] \odot y = 1$, we have $y \odot (y^* \odot x) \leq y$. If $z \leq x$ and $z \leq y$, then $z^* \oplus x = 1$ and $z^* \oplus y = 1$. It implies, by Theorem 3.1(8),

$$z^* \oplus [y \odot (y^* \odot x)] = (z^* \odot y) \odot [(z \odot y^*) \oplus (z^* \odot x)]$$

$$= 1 \odot (0 \oplus 1) = 1 \odot 1 = 1.$$
Thus, $z \leq y \odot (y^* \oplus x) = x \odot (x^* \oplus y)$. Hence, $x \land y = x \odot (x^* \oplus y)$.

Since $x \odot (x^* \oplus y) = y \odot (y^* \oplus x)$, we have $x \land y = y \land z$.

(3) $x \leq y$ iff $1 = x^* \oplus y$ iff $1 = y \odot x^*$ iff $y^* \leq x^*$.

(4) By (2) and (3), since $x^* \land y^* \leq x^*$, $y^*$ implies $x, y \leq (x^* \land y^*)^*$. Thus, $x \lor y \leq (x^* \land y^*)^*$. If $x, y \leq z$, then $z^* \leq x^* \land y^*$ implies $(x^* \land y^*)^* \leq z$. Hence

$$x \lor y = (x^* \land y^*)^* = x \odot (x^* \land y) = y \oplus (y^* \odot x) = y \lor x.$$ 

Therefore $(L, \leq, \land, \lor, *)$ is a lattice.

(5) From (2) and Theorem 3.1(8),

$$x \odot (y \land z) = x \odot [y \odot (y^* \oplus z)]$$
$$= (x \odot y) \odot [(x^* \odot y^*) \oplus (x \oplus z)]$$
$$= (x \odot y) \land (x \oplus z).$$

Let $y \leq z$. Then

$$x \odot y = x \odot (y \land z) = (x \odot y) \land (x \oplus z).$$

Hence $x \odot y \leq x \odot z$.

(R2) Let $x \leq y$. From (2),

$$(x \odot z)^* = x^* \odot z^* \geq y^* \odot z^* = (y \odot z)^*.$$ 

Hence $x \odot z \leq y \odot z$.

(6) We show that $x \odot y \leq x \land y \leq x \lor y \leq x \odot y$.

Since $x \odot y \leq x \odot 1 \leq x$ and $x \odot y \leq 1 \odot y \leq y$, we have $x \odot y \leq x \land y$.

Since $x = x \odot 0 \leq x \odot y$ and $y = 0 \odot y \leq x \odot y$, we have $x \lor y \leq x \odot y$.

(R3) (Galois correspondence): $(x \odot y) \leq z$ iff $x \leq (y \rightarrow z)$.

Let $(x \odot y) \leq z$. Then

$$1 = (x \odot y)^* \oplus z$$
$$= (x^* \odot y^*) \oplus z$$
$$= x^* \odot (y^* \oplus z)$$
$$= x^* \odot (y \rightarrow z).$$
Thus, \( x \leq (y \to z) \).
Let \( x \leq y \to z \). Then

\[
x \odot y = y \odot x \\
\leq y \odot (y \to z) \\
= y \odot (y^* \oplus z) \\
= y \land z \leq z.
\]

(B1) It is trivial from (2).

(7) Since \( x^* \oplus y = y \oplus x^* \), we have \( x \to y = y^* \to x^* \).

(B2)

\[
x \lor y = (x^* \land y^*)^* \\
= x \oplus (x^* \odot y) \\
= x^* \to (x^* \to y^*)^* \\
= (x^* \to y^*) \to x \quad \text{(by (7))} \\
= (y \to x) \to x.
\]

Since \((y \to x) \to x = x \oplus (x^* \odot y) = y \oplus (y^* \odot x) = (x \to y) \to y \) from Theorem 3.1(7), we have

\[
x \lor y = [(x \to y) \to y] \land [(y \to x) \to x].
\]

(B3) We will show that \((x \to y) \lor (y \to x) = 1 \).

(a) \( x \to (y \to z) = x^* \oplus (y^* \oplus z) = y^* \oplus (x^* \oplus z) = x \to y \to z \).

(b) \( (x \lor y) \to x = (x \lor y)^* \oplus x = (x^* \oplus x) \land (y^* \oplus x) = y^* \oplus x = y \to x \)

from (5). Similarly, \((x \lor y) \to y = x \to y \).

Since

\[
(y \to x) \to (x \to y) = [(x \lor y) \to x] \to [(x \lor y) \to y] \quad \text{(by (b))} \\
= [x^* \to (x \lor y)^*] \to [y^* \to (x \lor y)^*](\text{by (7))} \\
= y^* \to \{[x^* \to (x \lor y)^*] \to (x \lor y)^*\}(\text{by (a)}) \\
= y^* \to [x^* \lor (x \lor y)^*] \\
= [x^* \lor (x \lor y)^*]^* \to y \\
= [x \land (x \lor y)] \to y \\
= x \to y \\
= x^* \oplus y,
\]
we have
\[(x \rightarrow y) \lor (y \rightarrow x) = [(y \rightarrow x) \rightarrow (x \rightarrow y)] \rightarrow (x \rightarrow y)\]
\[= (x^* \oplus y) \rightarrow (x^* \oplus y)\]
\[= (x^* \oplus y)^* \oplus (x^* \oplus y)\]
\[= 1.\]

Hence \((L, \leq, \wedge, \lor, \ominus, \rightarrow, 0, 1)\) is an MV-algebra.

We can obtain the following corollary from Theorem 3.3.

**Corollary 3.4.** Let \(([0, 1], \otimes)\) be a t-norm and \(([0, 1], \sqcup)\) a t-conorm which satisfies the following conditions:

1. \(x^{**} = x\), and \(1^{*} = 0\).
2. \((x \sqcup y)^* = x^* \otimes y^*\).
3. \(x \sqcup x^* = 1\).
4. \(x \sqcup 0 = x\).
5. \(x \otimes (x^* \sqcup y) = y \otimes (y^* \sqcup x)\).
6. \(x \sqcup [y \otimes (y^* \sqcup z)] = (x \sqcup y) \otimes [(x^* \otimes y^*) \sqcup (x \otimes z)]\).

Then \(([0, 1], \leq, \wedge, \lor, \ominus, \rightarrow, 0, 1)\) is an MV-algebra.

**References**


Department of Mathematics
Kangnung National University
Kangnung, Kangwondo, 210-702