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SOME PROPERTIES OF MV-ALGEBRAS

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Abstract. In this paper, we obtain an algebraic structure which is equivalent to an MV-algebra. Moreover, we show that t-norm and t-conorm can be obtained from MV-algebras.

1. Introduction

Ward and Dilworth [9] introduced residuated lattices as the foundation of the algebraic structures of fuzzy logics. Hájeck [1] introduced a BL-algebra which is a general tool of a fuzzy logic. Recently, Höhle [2,3] extended the fuzzy set $f : X \to L$ where L is an MV-algebra in stead of an unit interval I or a lattice L.

In this paper, we obtain an algebraic structure which is equivalent to an MV-algebra. Moreover, we show that t-norm and t-conorm can be obtained from MV-algebras.

2. Preliminaries

Definition 2.1 ([3,8]). A lattice $(L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ is called a *residuated lattice* if it satisfies the following conditions: for each $x, y, z \in L$,

(R1) $(L, \odot, 1)$ is a commutative monoid,

(R2) if $x \leq y$, then $x \odot z \leq y \odot z$ ($\odot$ is an isotone operation),

(R3) (Galois correspondence): $(x \odot y) \leq z$ iff $x \leq (y \to z)$.

In a residuated lattice L, $x^* = (x \to 0)$ is called *complement* of $x \in L$.

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Definition 2.2 ([3,8]). A residuated lattice $(L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a *BL-algebra* if it satisfies the following conditions: for each $x, y \in L$,

(B1) $x \wedge y = x \odot (x \rightarrow y)$,
(B2) $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$,
(B3) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

A BL-algebra L is called an *MV-algebra* if $x = x^{**}$ for each $x \in L$.

Lemma 2.3 ([3,8]). *Let L be an MV-algebra. For $x, y, z \in L$, we have the following properties:*

(1) $x = (1 \rightarrow x)$,
(2) $1 = (x \rightarrow x)$,
(3) $x \leq y$ *iff* $1 = (x \rightarrow y)$,
(4) $x = y$ *iff* $1 = (x \rightarrow y) = (y \rightarrow x)$,
(5) *if* $y \leq z$, $(x \rightarrow y) \leq (x \rightarrow z)$,
(6) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
(7) $x \odot y = (x \rightarrow y^*)^*$,
(8) $x \leq y$ *iff* $x^* \geq y^*$,
(9) $x \rightarrow y = y^* \rightarrow x^*$.

Definition 2.4 ([10]). A binary operation $\otimes : [0,1] \times [0,1] \rightarrow [0,1]$ is called a *t-norm* if it satisfies the following conditions: for each $x, y, z \in L$,

(1) $x \otimes y = y \otimes x$,
(2) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$,
(3) $x \otimes 1 = x$,
(4) if $x \leq y$, $x \otimes z \leq y \otimes z$.

We define the t-conorm as a dual sense of t-norm.

Definition 2.5 ([10]). A binary operation $\uplus : [0,1] \times [0,1] \rightarrow [0,1]$ is called a *t-conorm* if it satisfies the following conditions: for each $x, y, z \in L$,

(1) $x \uplus y = y \uplus x$,
(2) $x \uplus (y \uplus z) = (x \uplus y) \uplus z$,
(3) $x \uplus 0 = x$,
(4) if $x \leq y$, $x \uplus z \leq y \uplus z$.

3. Some properties of MV-algebras

THEOREM 3.1. *Let* $(L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *be an MV-algebra. Define* $x \oplus y = x^* \rightarrow y$. *For each* $x, y, z \in L$, *we have the following properties.*

(1) $x^{**} = x$, $1^* = 0$.
(2) $(x \oplus y)^* = x^* \odot y^*$, $(x \odot y)^* = x^* \oplus y^*$.
(3) $x \oplus y = y \oplus x$, $x \odot y = y \odot x$.
(4) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, $x \odot (y \odot z) = (x \odot y) \odot z$.
(5) $x \oplus x^* = 1$, $x \odot x^* = 0$.
(6) $x \oplus 0 = x$, $x \odot 1 = x$.
(7) $x \odot (x^* \oplus y) = y \odot (y^* \oplus x)$, $x \oplus (x^* \odot y) = y \oplus (y^* \odot x)$.
(8) $x \oplus [y \odot (y^* \oplus z)] = (x \oplus y) \odot [(x^* \odot y^*) \oplus (x \oplus z)]$.
(9) *if* $y \leq z$, *then* $x \oplus y \leq x \oplus z$.

Proof. (1) Since $1 \rightarrow 0 = 0$ from Lemma 2.3(1), $1^* = 0$.

(2) Put $z = 0$ from Lemma 2.3(6). Then $(x \odot y)^* = x \rightarrow y^* = x^* \oplus y^*$. Furthermore, $(x \oplus y)^* = (x^* \rightarrow y)^* = x^* \odot y^*$ from Lemma 2.3(7).

(3-4) Since $(L, \odot)$ is a commutative monoid, that is, $x \odot y = y \odot x$ and $x \odot (y \odot z) = (x \odot y) \odot z$, by (2), $x \oplus y = y \oplus x$ and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

(5) By (B1), $0 = x \wedge 0 = x \odot (x \rightarrow 0) = x \odot x^*$. It implies $x \oplus x^* = 1$.

(6) Put $y = 1$ from Lemma 2.3(7). Then $x \odot 1 = (x \rightarrow 1^*)^* = x^{**} = x$. Moreover, by (2), $x \oplus 0 = x$.

(7) By (B1), $x \odot (x^* \oplus y) = x \odot (x \rightarrow y) = x \wedge y = y \wedge x = y \odot (y^* \oplus x)$. By (2), trivially, $x \oplus (x^* \odot y) = y \oplus (y^* \odot x)$.

(8) If $y \leq z$, by Lemma 2.3(8), $y^* \geq z^*$. By (R2), $x^* \odot y^* \geq x^* \odot z^*$. By Lemma 2.3(7-8), $(x^* \rightarrow y)^* \geq (x^* \rightarrow z)^*$ implies $x^* \rightarrow y \leq x^* \rightarrow z$. Since $y \wedge z \leq y, z$, we have

$$x^* \rightarrow (y \wedge z) \leq (x^* \rightarrow y) \wedge (x^* \rightarrow z).$$

Since $(x^* \rightarrow y) \wedge (x^* \rightarrow z) \leq (x^* \rightarrow y), (x^* \rightarrow z)$, by (R3), $x^* \odot ((x^* \rightarrow y) \wedge (x^* \rightarrow z)) \leq y, z$. Then $x^* \odot ((x^* \rightarrow y) \wedge (x^* \rightarrow z)) \leq y \wedge z$. It implies $(x^* \rightarrow y) \wedge (x^* \rightarrow z) \leq x^* \rightarrow (y \wedge z)$. So,

$$(x^* \rightarrow y) \wedge (x^* \rightarrow z) = x^* \rightarrow (y \wedge z).$$

Thus,

$$(x \oplus y) \wedge (x \oplus z) = x \oplus (y \wedge z).$$

Since $x \wedge y = x \odot (x \to y)$ from (B1), we obtain

$$x \oplus [y \odot (y^* \oplus z)] = (x \oplus y) \odot [(x^* \odot y^*) \oplus (x \oplus z)].$$

(9) Let $y \le z$. By (2) and (8) ,

$$x \oplus y = x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z).$$

Hence, $x \oplus y \le x \oplus z$. □

We can obtain the following corollary from Theorem 3.1.

COROLLARY 3.2. *If* $([0,1], \le, \wedge, \vee, \odot, \to, 0, 1)$ *is an MV-algebra, then* $([0,1], \odot)$ *is a t-norm and* $([0,1], \oplus)$ *is a t-conorm.*

THEOREM 3.3. *Let* $(L, \odot, \oplus, ^*, 0, 1)$ *be an algebraic structure which satisfies* (1)-(8) *in* Theorem 3.1. *Define*

$$x \le y \quad \textit{iff} \quad x^* \oplus y = 1$$
$$x \to y = x^* \oplus y$$

Then $(L, \le, \wedge, \vee, \odot, \to, 0, 1)$ *is an MV-algebra.*

Proof. (1) $(L, \le)$ is a partially ordered set.
(reflexive) Since $x^* \oplus x = 1$, $x \le x$.
(transitive) If $x \le y$ and $y \le z$, then $x^* \oplus y = 1$ and $y^* \oplus z = 1$. Since $0 \le 1$, $1 = 0^* \oplus 1 = 1 \oplus 1$. It implies $(x^* \oplus y) \oplus (y^* \oplus z) = x^* \oplus z = 1$. Thus $x \le z$.
(anti-symmetric) If $x \le y$ and $y \le x$, then $x^* \oplus y = 1$ and $y^* \oplus x = 1$. By Theorem 3.1(7), we have

$$x = x \odot 1 = x \odot (x^* \oplus y) = y \odot (y^* \oplus x) = y \odot 1 = y.$$

(2) We will show that $x \wedge y = x \odot (x^* \oplus y)$.
Since $[x \odot (x^* \oplus y)]^* \oplus x = [x^* \oplus (x \odot y^*)] \oplus x = (x^* \oplus x) \oplus (x \odot y^*)] = 1 \oplus (x \odot y^*) = 1$ because $0 \le (x \odot y^*)$, we have $x \odot (x^* \oplus y) \le x$. Since $[y \odot (y^* \oplus x)]^* \oplus y = [y^* \oplus (y \odot x^*)] \oplus y = 1$, we have $y \odot (y^* \oplus x) \le y$. If $z \le x$ and $z \le y$, then $z^* \oplus x = 1$ and $z^* \oplus y = 1$. It implies, by Theorem 3.1(8),

$$\begin{aligned} z^* \oplus [y \odot (y^* \oplus x)] &= (z^* \oplus y) \odot [(z \odot y^*) \oplus (z^* \oplus x)] \\ &= 1 \odot (0 \oplus 1) = 1 \odot 1 = 1. \end{aligned}$$

Thus, $z \leq y \odot (y^* \oplus x) = x \odot (x^* \oplus y)$. Hence, $x \wedge y = x \odot (x^* \oplus y)$.

Since $x \odot (x^* \oplus y) = y \odot (y^* \oplus x)$, we have $x \wedge y = y \wedge z$.

(3) $x \leq y$ iff $1 = x^* \oplus y$ iff $1 = y \oplus x^*$ iff $y^* \leq x^*$.

(4) By (2) and (3), since $x^* \wedge y^* \leq x^*, y^*$ implies $x, y \leq (x^* \wedge y^*)^*$. Thus, $x \vee y \leq (x^* \wedge y^*)^*$. If $x, y \leq z$, then $z^* \leq x^* \wedge y^*$ implies $(x^* \wedge y^*)^* \leq z$. Hence

$$x \vee y = (x^* \wedge y^*)^* = x \oplus (x^* \odot y) = y \oplus (y^* \odot x) = y \vee x.$$

Therefore $(L, \leq, \wedge, \vee, ^*)$ is a lattice.

(5) From (2) and Theorem 3.1(8) ,

$$\begin{aligned} x \oplus (y \wedge z) &= x \oplus [y \odot (y^* \oplus z)] \\ &= (x \oplus y) \odot [(x^* \odot y^*) \oplus (x \oplus z)] \\ &= (x \oplus y) \wedge (x \oplus z). \end{aligned}$$

Let $y \leq z$. Then

$$x \oplus y = x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z).$$

Hence $x \oplus y \leq x \oplus z$.

(R2) Let $x \leq y$. From (2),

$$(x \odot z)^* = x^* \oplus z^* \geq y^* \oplus z^* = (y \odot z)^*.$$

Hence $x \odot z \leq y \odot z$.

(6) We show that $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.

Since $x \odot y \leq x \odot 1 \leq x$ and $x \odot y \leq 1 \odot y \leq y$, we have $x \odot y \leq x \wedge y$. Since $x = x \oplus 0 \leq x \oplus y$ and $y = 0 \oplus y \leq x \oplus y$, we have $x \vee y \leq x \oplus y$.

(R3) (Galois correspondence): $(x \odot y) \leq z$ iff $x \leq (y \to z)$.

Let $(x \odot y) \leq z$. Then

$$\begin{aligned} 1 &= (x \odot y)^* \oplus z \\ &= (x^* \oplus y^*) \oplus z \\ &= x^* \oplus (y^* \oplus z) \\ &= x^* \oplus (y \to z). \end{aligned}$$

Thus, $x \leq (y \to z)$.

Let $x \leq y \to z$. Then

$$\begin{aligned} x \odot y &= y \odot x \\ &\leq y \odot (y \to z) \\ &= y \odot (y^* \oplus z) \\ &= y \wedge z \leq z. \end{aligned}$$

(B1) It is trivial from (2).

(7) Since $x^* \oplus y = y \oplus x^*$, we have $x \to y = y^* \to x^*$.

(B2)

$$\begin{aligned} x \vee y &= (x^* \wedge y^*)^* \\ &= x \oplus (x^* \odot y) \\ &= x^* \to (x^* \to y^*)^* \\ &= (x^* \to y^*) \to x \quad \text{(by (7))} \\ &= (y \to x) \to x. \end{aligned}$$

Since $(y \to x) \to x = x \oplus (x^* \odot y) = y \oplus (y^* \odot x) = (x \to y) \to y$ from Theorem 3.1(7), we have

$$x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x].$$

(B3) We will show that $(x \to y) \vee (y \to x) = 1$.

(a) $x \to (y \to z) = x^* \oplus (y^* \oplus z) = y^* \oplus (x^* \oplus z) = y \to (x \to z)$.

(b) $(x \vee y) \to x = (x \vee y)^* \oplus x = (x^* \oplus x) \wedge (y^* \oplus x) = y^* \oplus x = y \to x$ from (5). Similarly, $(x \vee y) \to y = x \to y$.

Since

$$\begin{aligned} (y \to x) \to (x \to y) &= [(x \vee y) \to x] \to [(x \vee y) \to y] \ \text{ (by (b))} \\ &= [x^* \to (x \vee y)^*] \to [y^* \to (x \vee y)^*] \text{(by (7))} \\ &= y^* \to \{[x^* \to (x \vee y)^*] \to (x \vee y)^*\} \text{(by (a))} \\ &= y^* \to [x^* \vee (x \vee y)^*] \\ &= [x^* \vee (x \vee y)^*]^* \to y \\ &= [x \wedge (x \vee y)] \to y \\ &= x \to y \\ &= x^* \oplus y, \end{aligned}$$

we have

$$\begin{aligned}(x \to y) \vee (y \to x) &= [(y \to x) \to (x \to y)] \to (x \to y)\\ &= (x^* \oplus y) \to (x^* \oplus y)\\ &= (x^* \oplus y)^* \oplus (x^* \oplus y)\\ &= 1.\end{aligned}$$

Hence $(L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ is an MV-algebra. □

We can obtain the following corollary from Theorem 3.3.

COROLLARY 3.4. *Let* $([0,1], \otimes)$ *be a t-norm and* $([0,1], \uplus)$ *a t-conorm which satisfies the following conditions:*

(1) $x^{**} = x$, *and* $1^* = 0$.
(2) $(x \uplus y)^* = x^* \otimes y^*$.
(3) $x \uplus x^* = 1$.
(4) $x \uplus 0 = x$.
(5) $x \otimes (x^* \uplus y) = y \otimes (y^* \uplus x)$.
(6) $x \uplus [y \otimes (y^* \uplus z)] = (x \uplus y) \otimes [(x^* \otimes y^*) \uplus (x \otimes z)]$.

Then $([0,1], \leq, \wedge, \vee, \otimes, \to, 0, 1)$ *is an MV-algebra.*

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