Kangweon-Kyungki Math. Jour. **10** (2002), No. 1, pp. 45–51

HEWITT REALCOMPACTIFICATIONS OF MINIMAL QUASI-F COVERS

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Abstract. Observing that a realcompactification Y of a space X is Wallman if and only if for any non-empty zero-set Z in Y, $Z \cap Y \neq \emptyset$, we will show that for any pseudo-Lindelöf space X, the minimal quasi-F $QF(\upsilon X)$ of υX is Wallman and that if X is weakly Lindelöf, then $QF(\upsilon X) = \upsilon$QF(X).

1. Introduction.

All spaces in this paper are Tychonoff spaces and $(\beta X, \beta_X)$, $((\upsilon X, \upsilon_X)$, resp.) denotes the Stone-$\check{C}$ech compactification (Hewitt realcompactification,resp.) of a space X. In [4],the minimal quasi-F cover $QF(\upsilon X)$ of a compact space X is constucted as an inverse limit space and in [10], Vermeer construct the minimal quasi-F cover of arbitrary Tychonoff spaces. Henriksen, Vermeer and Woods showed that for any weakly Lindelöf space X, βQF(X) and $QF(\beta X)$ are homeomorphic([6]).

In this paper, we first show that a realcompactification Y of a space X is Wallman if and only if for any non-empty zero-set Z in Y, $Z \cap Y \neq \emptyset$ and show that if X is a pseudo-Lindelöf space, then the minimal quasi-F cover $QF(\upsilon X)$ is a Wallman realcompactification of some cover of X. Finally, we will show that if X is weakly Lindelöf and pseudo-Lindelöf, then υQF(X) and $QF(\upsilon X)$ are homeomorphic and $QF(X)$ is pseudo-Lindelöf. For the terminology, we refer to [5] and [8].

2. Wallman realcompactification.

Recall that a pair (Y, j) or simply Y is called a compactification (realcompactification, resp.) of a space X if $j : X \hookrightarrow Y$ is a dence embedding and Y is a compact (realcompact, resp.) space. For any space

Received January 28, 2002.
2000 Mathematics Subject Classification: 54C10, 54D80, 54G05.
Key words and phrases: Covering map, quasi-F space, minimal cover.

X, let $C(X)(C^*(X)$,resp.) denote the ring of real-valued continuous functions (bounded functions,resp.) on X. A subspace S of a space X is said to be C-embedded ($C^* - embedded, resp.$) in X if every function in $C(S)(C^*(S)$,resp.) extends to a function in $C(X)(C^*(X)$,resp.).

Definition 2.1 ([9]). *Let X be a space and $\mathcal{F}$ a family of closed sets in X.Then $\mathcal{F}$ is called a separating nest generated intersection ring on X if*
(i) for each closed set H in X and $x \notin H$,there are disjoint sets in $\mathcal{F}$,one containing H and the other containing x;
(ii) it is closed under finite unions and countable intersections;
and
(iii) for any $F \in \mathcal{F}$, there are sequences (F_n) and (H_n) in $\mathcal{F}$ such that for any $n \in \mathbb{N}$, $X \backslash H_{n+1} \subset F_{n+1} \subset X \backslash H_n \subset F_n$ and $F = \cap F_n$.

For a space X, $Z(X)$ denotes the set of zero-sets in X, $\mathcal{L}(X)$ the set of seperating nest generated intersection rings on X and for any subspaces S of X and $\mathcal{F} \subset 2^X$,let $\mathcal{F}_S = \{F \cap S : F \in \mathcal{F}\}$. For a subspace S of a space X and $\mathcal{F} \in \mathcal{L}(X)$, $Z(X) \in \mathcal{L}(X)$ and $\mathcal{F}_S \in \mathcal{L}(S)$ ([9]).

Let X be a space and $\mathcal{F} \in \mathcal{L}(X)$. Then $\mathcal{F}$ is a normal base on X. Let $(\omega(X, F), \omega_X)$ be the Wallman compactification of X associated with $\mathcal{F}$ ([1]). Then $\mathcal{F} = Z(\omega(X, \mathcal{F}))_X$ and if (Y, j) is a compactification of X such that $\mathcal{F} = Z(Y)_X$, then there is a continuous map $f : \omega(X, \mathcal{F}) \to Y$ with $f \circ \omega_X = j$ ([9]).

Let $\upsilon(X, \mathcal{F})$=$\{\alpha$: α is an $\mathcal{F}$-ultrafilter on X with the countable intersection property$\}$.Then the topology on $\upsilon(X, \mathcal{F})$, taking sets of the form F^*=$\{\alpha \in \upsilon(X, \mathcal{F}) : F \in \alpha \}$ as a base for the the closed sets, coincides with the subspace topology on $\upsilon(X, \mathcal{F})$ of $\omega(X, \mathcal{F})$,$\upsilon(X, \mathcal{F})$ is a realcompactification of X (called *Wallman realcompactification*) ([9]), $\upsilon(X, \mathcal{F}) = \upsilon(X, \mathcal{F}^t)$ and $\omega(X, \mathcal{F}^t) = \beta(\upsilon(X, \mathcal{F}^t))$, where $\mathcal{F}^t = Z(\upsilon(X, \mathcal{F}))_X$ ([3]).

In a space (X, τ), the family of G_δ-sets on X forms a base for a topology τ_δ on X and for $A \in X$, $\aleph_1 - cl_X(A)$ denotes the closure of A in (X, τ_δ).

Theorem 2.2. *A realcompactification (Y, j) of a space X is Wallman if and only if for non-empty zero-set Z in Y, $Z \cap X \neq \emptyset$. In this case, $Y = \upsilon(X, \mathcal{F})$ and $\mathcal{F} = Z(Y)_X$.*

Proof. ($\Rightarrow$) Since Y is a Wallman realcompactification of X, $Y = \upsilon(X, \mathcal{G})$ for some $\mathcal{G} \in \mathcal{L}(X)$. Then $\upsilon(X, \mathcal{G}) = \upsilon(X, \mathcal{G}^t)$ and $\beta(\upsilon(X, \mathcal{G}^t)) = \omega(X, \mathcal{G}^t)$, where $\mathcal{G}^t = Z(\upsilon(X, \mathcal{G}))_X$ ([3]).Hence there is a continuous map $f : \omega(X, \mathcal{G}^t) \to \omega(X, \mathcal{G})$ with $f \circ l = k \circ h$, where $h : \upsilon(X, \mathcal{G}^t) \to \upsilon(X, \mathcal{G})$ is a homeomorphism and $l : \upsilon(X, \mathcal{G}^t) \hookrightarrow \omega(X, \mathcal{G}^t)$ and $k : \upsilon(X, \mathcal{G}) \hookrightarrow \omega(X, \mathcal{G})$ are dense embeddings. Take any non-empty zero-set Z in Y. Since $h^{-1}(Z)$ is a zero-set in $\upsilon(X, \mathcal{G}^t)$, there is a zero-set A in $\beta(\upsilon(X, \mathcal{G}^t)) = \omega(X, \mathcal{G}^t)$ such that $h^{-1}(Z) = A \cap \upsilon(X, \mathcal{G}^t)$. Since $h^{-1}(Z) \neq \emptyset$,pick $\alpha \in A \cap \upsilon(X, \mathcal{G}^t)$. Then there is a countable family $\{Z_n : n \in \mathbb{N}\}$ of zero-set neighborhoods of α in $\omega(X, \mathcal{G}^t)$ such that $A = \bigcap Z_n$. For any $n \in \mathbb{N}$. $Z_n \cap X \in \mathcal{G}^t$ and hence $Z_n \cap X \in \alpha$. Since α has the countable intersection property, $A \cap X = (\bigcap Z_n) \cap X \neq \emptyset$. Thus $h^{-1}(Z) = Z \cap X \neq \emptyset$.

($\Leftarrow$) Let $\mathcal{F} = Z(Y)_X$, then $\mathcal{F} \in \mathcal{L}(X)$. Note that $Z(\beta Y)_X = Z(Y)_X = \mathcal{F}$. Hence,there is a continuous map $g : \omega(X, \mathcal{F}) \to \beta Y$ with $g \circ \omega_X = \beta Y \circ j$. Let A and B be zero-sets in $\omega(X, \mathcal{F})$ with $A \cap B \cap X = \emptyset$, then $A \cap X$, $B \cap X \in \mathcal{F}$.Hence there are C,D in$Z(Y)$ with $A \cap X = C \cap X$ and $B \cap X = D \cap X$. Since $C \cap D \cap X = \emptyset$ and $C \cap D \in Z(Y)$, $C \cap D = \emptyset$ and hence $cl_{\beta Y}(C) \cap cl_{\beta Y}(D) = \emptyset$. So $cl_{\beta Y}(A \cap X) \cap cl_{\beta Y}(B \cap X) = \emptyset$. By Urysohn's extension theorem, there is a continuous map $h : \beta Y \to \omega(X, \mathcal{F})$ such that $\omega_X = h \circ \beta_Y \circ j$ and so h is a homeomorphism.

Note that $\aleph_1 - cl_{\beta Y}(X) \subset \aleph_1 - cl_{\beta Y}(Y)$. Let $x \notin \aleph_1 - cl_{\beta Y}(X)$. Then there is a zero-set Z in βY such that $x \in Z$ and $Z \cap X = \emptyset$. Since $(S \cap Y) \cap X = \emptyset$, $Z \cap Y = \emptyset$. So $x \notin \aleph_1 - cl_{\beta Y}(Y)$. Hence $\aleph_1 - cl_{\beta Y}(X) = \aleph_1 - cl_{\beta Y}(Y)$. It is well-known that $\upsilon(X, \mathcal{F}) = \aleph_1 - cl_{\omega(X,\mathcal{F})}(X)$ ([1]). Since $\omega(X, \mathcal{F})$ and βY are homeomorphic, $\aleph_1 - cl_{\beta Y}(Y) = \upsilon(X, \mathcal{F})$ and since Y is a realcompact space, $\aleph_1 - cl_{\beta Y}(Y) = Y$. So $Y = \omega(X, \mathcal{F})$.$\square$

3. Quasi-F covers of Hewitt realcompactifications.

Recall that a space X is called pseudocompact if υX is compact. The following definition is a generalization of pseudocompact spaces.

Definition 3.1. *A space X is called pseudo-Lindelöf if υX is Lindelöf.*

It is well-known that for a paracompact (or separable) space X, X is pseudo-Lindelöf if and only if every separating nest generated intersection ring on X is complete ([3]).

PROPOSITION 3.2. *Let X be a space. Then X is pseudo-Lindelöf if and only if every Wallman realcompactification of X is Lindelöf.*

Proof. Suppose that X is pseudo-Lindelöf. Let (Y, j) be a Wallman realcompactification of X and $\mathcal{G}$ a z-filter on Y with the countable intersection property. Since Y is realcompact, there is a continuous map $h : \upsilon X \to Y$ such that $h \circ \upsilon_X = j$. By Theorem 1.2, for any $G \in \mathcal{G}$, $G \cap X \neq \emptyset$. Hence $\mathcal{G}_X = \{G \cap X : G \in \mathcal{G}\}$ is a z-filter on X with the countable intersection property. For any $G \in \mathcal{G}$, $cl_{\upsilon X}(G \cap X)$ is a zero-set in υX. So $\mathcal{F} = \{cl_{\upsilon X}(G \cap X) : G \in \mathcal{G}\}$ is a z-filter on υX. Since υX is Lindelöf, there is an $\alpha \in \upsilon X$ such that $\alpha \in \bigcap \mathcal{F}$. Hence for any $G \in \mathcal{G}$, $cl_{\upsilon X}(G \cap X) \in \alpha$ and

$$\begin{aligned} h(\alpha) &\in h(cl_{\upsilon X}(G \cap X)) \\ &\subseteq cl_Y(h(G \cap X)) \\ &= cl_Y(G \cap X) \\ &\subseteq cl_Y(G) \\ &= G. \end{aligned}$$

Thus $\bigcap \mathcal{G} \neq \emptyset$ and so Y is Lindelöf. The convers is trivial, because $\upsilon X = \upsilon(X, Z(X))$. □

DEFINITION 3.3. *A space X is called a quasi-F space if for any zero-sets A, B in X, $cl_X(int_X(A \cap B)) = cl_X(int_X(A)) \cap cl_X(int_X(B))$, equivalently, every dense cozero-set in X is $C^* -$ embdded in X.*

For any covering map (= compact, closed and irreducible) $\Phi_X : QF(X) \longrightarrow X$ such that for any quasi-F space Y and covering map $f : Y \longrightarrow X$, there is a covering map $g : Y \longrightarrow QF(X)$ with $\Phi_X \circ g = f$, that is, $(QF(X), \Phi_X)$ is the minimal quasi-F cover of F ([6]).

Recall that a space X is called *weakly Lindelöf* if for any open cover $\mathcal{U}$ of X, there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$, $\bigcup \mathcal{V}$ is dense in X. It is shown that for any weakly Lindelöf space X, $\beta QF(X)$ and $QF(\beta X)$ are homeomorphic ([6]) and that $(\Phi^{-1}(X), l)$ is the minimal quasi-F cover of X, where $(QF(\upsilon X), \Phi)$ is the minimal quasi-F cover of υX and $l : \Phi^{-1}(X) \longrightarrow X$ is the restriction and corestriction of Φ with respect to $\Phi^{-1}(X)$ and X, respectively ([7]).

Theorem 3.4. *Let X be a pseudo-Lindelöf space. Then $QF(\upsilon X)$ is a Wallman realcompactification of $\Phi^{-1}(X)$.*

Proof. Consider the following commutative diagram ;

$$\begin{array}{ccc} Y & \xrightarrow{\ l\ } & X \\ j\Big\downarrow & & \Big\downarrow \upsilon_X \\ QF(\upsilon X) & \xrightarrow{\ \Phi\ } & \upsilon X \end{array}$$

where j is the inclusion map and $Y = \Phi^{-1}(X) = QF(X)$. Since $QF(\upsilon X)$ is realcompact, there is a continuous map $h : \upsilon Y \longrightarrow QF(\upsilon X)$ such that $h \circ \upsilon_Y = j$. Since $QF(\upsilon X)$ is Lindelöf, $QF(\upsilon X)$ is weakly Lindelöf and hence $\beta QF(\upsilon X)$ and $QF(\beta X)$ are homeomorphic. Hence there is a continuous map $g : \beta Y \longrightarrow QF(\beta X)$ such that the following diagram ;

$$\begin{array}{ccc} \upsilon Y & \xrightarrow{\ h\ } & QF(\upsilon X) \\ \beta_{\upsilon Y}\Big\downarrow & & \Big\downarrow \beta_{QF(\upsilon X)} \\ \beta Y & \xrightarrow{\ g\ } & QF(\beta X) \end{array}$$

commutes. Since $g \circ \Phi_{\beta X} : \beta Y \longrightarrow \beta X$ is onto and $\Phi \circ h|_Y = l$ is perfect, h is onto ([9]). Take any non-empty zero-set Z in $QF(\upsilon X)$. Then $h^{-1}(Z)$ is non-empty zero-set in υY and hence $Z \cap X = h^{-1}(Z) \cap X \neq \emptyset$. By Theorem 1.2, $QF(\upsilon X)$ is Wallman. □

Corollary 3.5. *Let X be a weakly Lindelöf and pseudo-Lindelöf space. Then $QF(\upsilon X)$ and $\upsilon QF(X)$ are homeomorphic.*

Proof. Since $QF(\upsilon X)$ is a realcompact space, there is a continuous map $h : \upsilon QF(X) \longrightarrow QF(\upsilon X)$ such that $h \circ \upsilon_{QF(X)} = j$, where $j : QF(X) \longrightarrow QF(\upsilon X)$ is the inclusion map. Take any disjoint zero-sets A, B in $QF(X)$. Then there are disjoint zero-sets C, D in $QF(X)$ such that $A \subseteq int_{QF(X)}(C)$ and $B \subseteq int_{QF(X)}(D)$. Since X is weakly Lindelöf, $cl_{QF(X)}(int_{QF(X)}(C))$ and $cl_{QF(X)}(int_{QF(X)}(D))$ is weakly Lindelöf. Hence there is a zero-set E in $QF(\upsilon X)$ such that

$$cl_{QF(X)}(int_{QF(X)}(D)) \subseteq int_{QF(\upsilon X)}(E)$$

and

$$cl_{QF(X)}(int_{QF(X)}(C)) \cap int_{QF(\upsilon X)}(E) = \emptyset.$$

Since $QF(X)$ is a quasi-F space,

$$cl_{QF(X)}(int_{QF(X)}(C)) \cap cl_{QF(X)}(int_{QF(\upsilon X)}(E) \cap QF(X)) = \emptyset.$$

Similarly,there are a zero-set F in $QF(\upsilon X)$ such that

$$cl_{QF(X)}(int_{QF(X)}(C)) \subseteq int_{QF(\upsilon X)}(F)$$

and

$$cl_{QF(X)}(int_{QF(\upsilon X)}(E) \cap QF(X)) \cap int_{QF(\upsilon X)}(F) = \emptyset.$$

Since $QF(X)$ is dense in $QF(\upsilon X)$, $int_{QF(\upsilon X)}(E) \cap int_{QF(\upsilon X)}(F) = \emptyset$. Hense

$$cl_{QF(\upsilon X)}(int_{QF(\upsilon X)}(E)) \cap cl_{QF(\upsilon X)}(int_{QF(\upsilon X)}(F)) = \emptyset.$$

and so A and B are completely separated in $QF(\upsilon X)$. By Urysohn's extension theorem, $QF(X)$ is $C^* - embedded$ in $QF(\upsilon X)$. Take any zero-set Z in $QF(\upsilon X)$ such that $Z \cap QF(X) = \emptyset$. By the above theorem, $Z = \emptyset$. Hence Z and $QF(X)$ are completely separated in $QF(\upsilon X)$ and so $QF(X)$ is $C - embedded$ in $QF(\upsilon X)$ ([5]). Since $\upsilon QF(X)$ is the unique realcompactification of $QF(X)$ which is $C - embedded$ in it, $\upsilon QF(X)$ and $QF(\upsilon X)$ are homeomorphic. □

By Corollary 3.5, we have the following :

Corollary 3.6. *If X is a weakly Lindelöf pseudo-compact space, then $QF(X)$ is pseudo-Lindelöf.*

References

[1] R.A. Alo and H.L. Shapiro, *Z-realcompactifications and normal bases*, J. Austral. Math. Soc. **9** (1966), 489-495.

[2] R. Blair, *Spaces in which special sets are Z-embedded*, Canad. J. Math. **28** (1976), 673-690.

[3] J,L,Blasco, *Complete bases and Wallman realcompactifications*, Proc. Math. Soc. **75** (1979), 114-118.

[4] F. Dashiell, A. W. Hager and M. Henviksn, *Order-Cauchy completions of rings and vector lattices of continuous funcations*, Canad. J. Math. **32** (1980), 657-685.

[5] L. Gillman and M. Jerisn, *Rings of continuous functions*, Van Nostrand, Princeton, New York, 1960.

[6] M. Henriksen, J. Vermeer, and R. G. Woods, *Quasi-F-covers of Tychonoff spaces*, Tran. Amer. Math. Soc. **303** (1987), 779-804.

[7] C. I. Kim, *Minimal covers and filter spaces*, Topol. and its Appl. **72** (1996), 31-37.

[8] J. Porter and R. G. Woods, *Extensions and Absolutes of Hausdorff Spaces*, Springer,Berlin, 1988.

[9] A. K. Steiner and E. F. Steiner, *Nest generated intersection ring in Tychonoff spaces*, Trans. Amer. Math. Soc. **148** (1970), 589-601.

[10] J. Vermeer, *On perfect irreducible preimage*, Topology Proc. **9** (1984), 173-189.

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