HEWITT REALCOMPACTIFICATIONS
OF MINIMAL QUASI-$F$ COVERS

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Abstract. Observing that a realcompactification $Y$ of a space $X$ is Wallman if and only if for any non-empty zero-set $Z$ in $Y$, $Z \cap Y \neq \emptyset$, we will show that for any pseudo-Lindelöf space $X$, the minimal quasi-$F$ $QF(\upsilon X)$ of $\upsilon X$ is Wallman and that if $X$ is weakly Lindelöf, then $QF(\upsilon X) = \upsilon QF(X)$.

1. Introduction.
All spaces in this paper are Tychonoff spaces and $(\beta X, \beta_X)$, $(\upsilon X, \upsilon_X)$, resp.) denotes the Stone-Čech compactification (Hewitt realcompactification, resp.) of a space $X$. In [4], the minimal quasi-$F$ cover $QF(\upsilon X)$ of a compact space $X$ is constructed as an inverse limit space and in [10], Vermeer constructs the minimal quasi-$F$ cover of arbitrary Tychonoff spaces. Henriksen, Vermeer and Woods showed that for any weakly Lindelöf space $X$, $\beta QF(X)$ and $QF(\beta X)$ are homeomorphic([6]).

In this paper, we first show that a realcompactification $Y$ of a space $X$ is Wallman if and only if for any non-empty zero-set $Z$ in $Y$, $Z \cap Y \neq \emptyset$ and show that if $X$ is a pseudo-Lindelöf space, then the minimal quasi-$F$ cover $QF(\upsilon X)$ is a Wallman realcompactification of some cover of $X$. Finally, we will show that if $X$ is weakly Lindelöf and pseudo-Lindelöf, then $\upsilon QF(X)$ and $QF(\upsilon X)$ are homeomorphic and $QF(X)$ is pseudo-Lindelöf. For the terminology, we refer to [5] and [8].

2. Wallman realcompactification.
Recall that a pair $(Y, j)$ or simply $Y$ is called a compactification (realcompactification, resp.) of a space $X$ if $j : X \hookrightarrow Y$ is a dense embedding and $Y$ is a compact (realcompact, resp.) space. For any space...
Let $X$, let $C(X)(C^*(X),\text{resp.})$ denote the ring of real-valued continuous functions (bounded functions, resp.) on $X$. A subspace $S$ of a space $X$ is said to be $C$-embedded ($C^*$-embedded, resp.) in $X$ if every function in $C(S)(C^*(S),\text{resp.})$ extends to a function in $C(X)(C^*(X),\text{resp.})$.

**Definition 2.1 ([9])** Let $X$ be a space and $\mathcal{F}$ a family of closed sets in $X$. Then $\mathcal{F}$ is called a separating nest generated intersection ring on $X$ if

(i) for each closed set $H$ in $X$ and $x \notin H$, there are disjoint sets in $\mathcal{F}$, one containing $H$ and the other containing $x$;

(ii) it is closed under finite unions and countable intersections;

and

(iii) for any $F \in \mathcal{F}$, there are sequences $(F_n)$ and $(H_n)$ in $\mathcal{F}$ such that for any $n \in \mathbb{N}$, $X \setminus H_{n+1} \subset F_{n+1} \subset X \setminus H_n \subset F_n$ and $F = \cap F_n$.

For a space $X$, $Z(X)$ denotes the set of zero-sets in $X$, $\mathcal{L}(X)$ the set of separating nest generated intersection rings on $X$ and for any subspaces $S$ of $X$ and $\mathcal{F} \subset 2^X$, let $\mathcal{F}_S = \{F \cap S : F \in \mathcal{F}\}$. For a subspace $S$ of a space $X$ and $\mathcal{F} \in \mathcal{L}(X)$, $Z(X) \in \mathcal{L}(X)$ and $\mathcal{F}_S \in \mathcal{L}(S)$ ([9]).

Let $X$ be a space and $\mathcal{F} \in \mathcal{L}(X)$. Then $\mathcal{F}$ is a normal base on $X$. Let $(\omega(X, \mathcal{F}), \omega_X)$ be the Wallman compactification of $X$ associated with $\mathcal{F}$ ([1]). Then $\mathcal{F} = Z(\omega(X, \mathcal{F}))_X$ and if $(Y, j)$ is a compactification of $X$ such that $\mathcal{F} = Z(Y)_X$, then there is a continuous map $f : \omega(X, \mathcal{F}) \to Y$ with $f \circ \omega_X = j$ ([9]).

Let $v(X, \mathcal{F}) = \{\alpha : \alpha$ is an $\mathcal{F}$-ultrafilter on $X$ with the countable intersection property\}. Then the topology on $v(X, \mathcal{F})$, taking sets of the form $F^\alpha = \{\alpha \in v(X, \mathcal{F}) : F \in \alpha\}$ as a base for the the closed sets, coincides with the subspace topology on $v(X, \mathcal{F})$ of $\omega(X, \mathcal{F}), v(X, \mathcal{F})$ is a realcompactification of $X$ (called Wallman realcompactification) ([9]), $v(X, \mathcal{F}) = v(X, \mathcal{F}^\text{t})$ and $\omega(X, \mathcal{F}^\text{t}) = \beta(v(X, \mathcal{F}^\text{t}))$, where $\mathcal{F}^\text{t} = Z(v(X, \mathcal{F}))_X$ ([3]).

In a space $(X, \tau)$, the family of $G_\delta$-sets on $X$ forms a base for a topology $\tau_\delta$ on $X$ and for $A \in X$, $\mathcal{R}_1 - cl_X(A)$ denotes the closure of $A$ in $(X, \tau_\delta)$.

**Theorem 2.2.** A realcompactification $(Y, j)$ of a space $X$ is Wallman if and only if for non-empty zero-set $Z$ in $Y$, $Z \cap X \neq \emptyset$. In this case, $Y = v(X, \mathcal{F})$ and $\mathcal{F} = Z(Y)_X$. 
Proof. \((\Rightarrow)\) Since \(Y\) is a Wallman realcompactification of \(X\), \(Y = v(X, G)\) for some \(G \in \mathcal{L}(X)\). Then \(v(X, G) = v(X, G')\) and \(\beta(v(X, G')) = \omega(X, G')\), where \(G' = Z(v(X, G))_X\) ([3]). Hence there is a continuous map \(f : \omega(X, G') \to \omega(X, G)\) with \(f \circ l = k \circ h\), where \(h : v(X, G') \to v(X, G)\) is a homeomorphism and \(l : v(X, G') \to \omega(X, G')\) and \(k : v(X, G) \to \omega(X, G)\) are dense embeddings. Take any non-empty zero-set \(Z\) in \(Y\). Since \(h^{-1}(Z)\) is a zero-set in \(v(X, G')\), there is a zero-set \(A\) in \(\beta(v(X, G')) = \omega(X, G')\) such that \(h^{-1}(Z) = A \cap v(X, G')\). Since \(h^{-1}(Z) \neq \emptyset\), pick \(\alpha \in A \cap v(X, G')\). Then there is a countable family \(\{Z_n : n \in \mathbb{N}\}\) of zero-set neighborhoods of \(\alpha\) in \(\omega(X, G')\) such that \(A = \bigcap Z_n\). For any \(n \in \mathbb{N}\), \(Z_n \cap X \in G\) and hence \(Z_n \cap X \in \alpha\). Since \(\alpha\) has the countable intersection property, \(A \cap X = (\bigcap Z_n) \cap X \neq \emptyset\). Thus \(h^{-1}(Z) = Z \cap X \neq \emptyset\).

\((\Leftarrow)\) Let \(F = Z(Y)_X\), then \(F \in \mathcal{L}(X)\). Note that \(Z(\beta Y)_X = Z(Y)_X = F\). Hence, there is a continuous map \(g : \omega(X, F) \to \beta Y\) with \(g \circ \omega_X = \beta Y \circ j\). Let \(A\) and \(B\) be zero-sets in \(\omega(X, F)\) with \(A \cap B \cap X = \emptyset\), then \(A \cap X, B \cap X \in F\). Hence there are \(C, D\) in \(Z(Y)\) with \(A \cap X = C \cap X\) and \(B \cap X = D \cap X\). Since \(C \cap D \cap X = \emptyset\) and \(C \cap D \in Z(Y)\), \(C \cap D = \emptyset\) and hence \(cl_{\beta Y}(C) \cap cl_{\beta Y}(D) = \emptyset\). So \(cl_{\beta Y}(A \cap X) \cap cl_{\beta Y}(B \cap X) = \emptyset\). By Urysohn's extension theorem, there is a continuous map \(h : \beta Y \to \omega(X, F)\) such that \(\omega_X = h \circ \beta Y \circ j\) and so \(h\) is a homeomorphism.

Note that \(\kappa_1 - cl_{\beta Y}(X) \subset \kappa_1 - cl_{\beta Y}(Y)\). Let \(x \notin \kappa_1 - cl_{\beta Y}(X)\). Then there is a zero-set \(Z\) in \(\beta Y\) such that \(x \in Z\) and \(Z \cap Y = \emptyset\). Since \((S \cap Y) \cap X = \emptyset\), \(Z \cap Y = \emptyset\). So \(x \notin \kappa_1 - cl_{\beta Y}(Y)\). Hence \(\kappa_1 - cl_{\beta Y}(X) = \kappa_1 - cl_{\beta Y}(Y)\). It is well-known that \(v(X, F) = \kappa_1 - cl_{\omega(X, F)}(X)\) ([1]). Since \(\omega(X, F)\) and \(\beta Y\) are homeomorphic, \(\kappa_1 - cl_{\beta Y}(Y) = v(X, F)\) and since \(Y\) is a realcompact space, \(\kappa_1 - cl_{\beta Y}(Y) = Y\). So \(Y = \omega(X, F)\).

\section{3. Quasi-\(F\) covers of Hewitt realcompactifications.}

Recall that a space \(X\) is called pseudocompact if \(vX\) is compact. The following definition is a generalization of pseudocompact spaces.

**Definition 3.1.** A space \(X\) is called pseudo-Lindelöf if \(vX\) is Lindelöf.

It is well-known that for a paracompact (or separable) space \(X\), \(X\) is pseudo-Lindelöf if and only if every separating nest generated...
intersection ring on $X$ is complete ([3]).

**Proposition 3.2.** Let $X$ be a space. Then $X$ is pseudo-Lindelöf if and only if every Wallman realcompactification of $X$ is Lindelöf.

**Proof.** Suppose that $X$ is pseudo-Lindelöf. Let $(Y, j)$ be a Wallman realcompactification of $X$ and $G$ a $z$-filter on $Y$ with the countable intersection property. Since $Y$ is realcompact, there is a continuous map $h : \nu X \rightarrow Y$ such that $h \circ \nu X = j$. By Theorem 1.2, for any $G \in G$, $G \cap X \neq \emptyset$. Hence $G_X = \{G \cap X : G \in G\}$ is a $z$-filter on $X$ with the countable intersection property. For any $G \in G$, $cl_{\nu X}(G \cap X)$ is a zero-set in $\nu X$. So $\mathcal{F} = \{cl_{\nu X}(G \cap X) : G \in G\}$ is a $z$-filter on $\nu X$. Since $\nu X$ is Lindelöf, there is an $\alpha \in \nu X$ such that $\alpha \in \bigcap \mathcal{F}$. Hence for any $G \in G$, $cl_{\nu X}(G \cap X) \subseteq cl_{\nu X}(\alpha)$ and $\Phi_{X}(G) = G$.

Thus $\bigcap G \neq \emptyset$ and so $Y$ is Lindelöf. The converse is trivial, because $\nu X = \nu(X, Z(X))$. □

**Definition 3.3.** A space $X$ is called a quasi-$F$ space if for any zero-sets $A, B$ in $X$, $cl_X(int_X(A \cap B)) = cl_X(int_X(A)) \cap cl_X(int_X(B))$, equivalently, every dense cozero-set in $X$ is $C^*-$embedded in $X$.

For any covering map (= compact, closed and irreducible) $\Phi_X : QF(X) \rightarrow X$ such that for any quasi-$F$ space $Y$ and covering map $f : Y \rightarrow X$, there is a covering map $g : Y \rightarrow QF(X)$ with $\Phi_X \circ g = f$, that is, $(QF(X), \Phi_X)$ is the minimal quasi-$F$ cover of $F$ ([6]).

Recall that a space $X$ is called weakly Lindelöf if for any open cover $\mathcal{U}$ of $X$, there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$, $\bigcup \mathcal{V}$ is dense in $X$. It is shown that for any weakly Lindelöf space $X$, $\beta QF(X)$ and $QF(\beta X)$ are homeomorphic ([6]) and that $(\Phi^{-1}(X), l)$ is the minimal quasi-$F$ cover of $X$, where $(QF(\nu X), \Phi)$ is the minimal quasi-$F$ cover of $\nu X$ and $l : \Phi^{-1}(X) \rightarrow X$ is the restriction and corestriction of $\Phi$ with respect to $\Phi^{-1}(X)$ and $X$, respectively ([7]).
Theorem 3.4. Let $X$ be a pseudo-Lindelöf space. Then $QF(vX)$ is a Wallman realcompactification of $\Phi^{-1}(X)$.

Proof. Consider the following commutative diagram:

$$
\begin{array}{ccc}
Y & \overset{l}{\longrightarrow} & X \\
\uparrow{j} & & \uparrow{v_X} \\
QF(vX) & \overset{\Phi}{\longrightarrow} & vX
\end{array}
$$

where $j$ is the inclusion map and $Y = \Phi^{-1}(X) = QF(X)$. Since $QF(vX)$ is realcompact, there is a continuous map $h : vY \longrightarrow QF(vX)$ such that $h \circ v_Y = j$. Since $QF(vX)$ is Lindelöf, $QF(vX)$ is weakly Lindelöf and hence $\beta QF(vX)$ and $QF(\beta X)$ are homeomorphic. Hence there is a continuous map $g : \beta Y \longrightarrow QF(\beta X)$ such that the following diagram:

$$
\begin{array}{ccc}
vY & \overset{h}{\longrightarrow} & QF(vX) \\
\downarrow{\beta v_Y} & & \downarrow{\beta_{QF(vX)}} \\
\beta Y & \overset{g}{\longrightarrow} & QF(\beta X)
\end{array}
$$

commutes. Since $g \circ \Phi_{\beta X} : \beta Y \longrightarrow \beta X$ is onto and $\Phi \circ h|_Y = l$ is perfect, $h$ is onto ([9]). Take any non-empty zero-set $Z$ in $QF(vX)$. Then $h^{-1}(Z)$ is non-empty zero-set in $vY$ and hence $Z \cap X = h^{-1}(Z) \cap X \neq \emptyset$. By Theorem 1.2, $QF(vX)$ is Wallman. □

Corollary 3.5. Let $X$ be a weakly Lindelöf and pseudo-Lindelöf space. Then $QF(vX)$ and $vQF(X)$ are homeomorphic.

Proof. Since $QF(vX)$ is a realcompact space, there is a continuous map $h : vQF(X) \longrightarrow QF(vX)$ such that $h \circ v_{QF(X)} = j$, where $j : QF(X) \longrightarrow QF(vX)$ is the inclusion map. Take any disjoint zero-sets $A, B$ in $QF(X)$. Then there are disjoint zero-sets $C, D$ in $QF(X)$ such that $A \subseteq int_{QF(X)}(C)$ and $B \subseteq int_{QF(X)}(D)$. Since $X$ is weakly Lindelöf, $cl_{QF(X)}(int_{QF(X)}(C))$ and $cl_{QF(X)}(int_{QF(X)}(D))$ is weakly Lindelöf. Hence there is a zero-set $E$ in $QF(vX)$ such that

$$
cl_{QF(X)}(int_{QF(X)}(E)) \subseteq int_{QF(vX)}(E)
$$
and
\[ \text{cl}_{QF(X)}(\text{int}_{QF(X)}(C)) \cap \text{int}_{QF(vX)}(E) = \emptyset. \]
Since \(QF(X)\) is a quasi-\(F\) space,
\[ \text{cl}_{QF(X)}(\text{int}_{QF(X)}(C)) \cap \text{int}_{QF(X)}(\text{int}_{QF(vX)}(E) \cap QF(X)) = \emptyset. \]
Similarly, there are a zero-set \(F\) in \(QF(vX)\) such that
\[ \text{cl}_{QF(X)}(\text{int}_{QF(X)}(C)) \subseteq \text{int}_{QF(vX)}(F) \]
and
\[ \text{cl}_{QF(X)}(\text{int}_{QF(vX)}(E) \cap QF(X)) \cap \text{int}_{QF(vX)}(F) = \emptyset. \]
Since \(QF(X)\) is dense in \(QF(vX)\), \(\text{int}_{QF(vX)}(E) \cap \text{int}_{QF(vX)}(F) = \emptyset\).
Hence
\[ \text{cl}_{QF(vX)}(\text{int}_{QF(vX)}(E)) \cap \text{cl}_{QF(vX)}(\text{int}_{QF(vX)}(F)) = \emptyset. \]
and so \(A\) and \(B\) are completely separated in \(QF(vX)\). By Urysohn’s extension theorem, \(QF(X)\) is \(C^* - \text{embedded}\) in \(QF(vX)\). Take any zero-set \(Z\) in \(QF(vX)\) such that \(Z \cap QF(X) = \emptyset\). By the above theorem, \(Z = \emptyset\). Hence \(Z\) and \(QF(X)\) are completely separated in \(QF(vX)\) and so \(QF(X)\) is \(C - \text{embedded}\) in \(QF(vX)\) ([5]). Since \(vQF(X)\) is the unique realcompactification of \(QF(X)\) which is \(C - \text{embedded}\) in it, \(vQF(X)\) and \(QF(vX)\) are homeomorphic. \(\square\)

By Corollary 3.5, we have the following:

**Corollary 3.6.** If \(X\) is a weakly Lindelöf pseudo-compact space, then \(QF(X)\) is pseudo-Lindelöf.

**References**


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