DOMAIN OF EXISTENCE OF A PERTURBED CAUCHY PROBLEM OF ODE

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ABSTRACT. In this paper we consider the Cauchy problem
\[ \frac{dx(t)}{dt} = x(t)^n, \quad x(0) = x_0. \]

The domain of existence is lower semicontinuous for perturbations of the data. We present a simple formula for the time of existence which is exact when there exists no perturbations.

1. Introduction

The simple example of the Cauchy problem
\[ \frac{dx(t)}{dt} = x(t)^n, \quad x(t) = x_0 > 0, \quad n \geq 2 \]
shows that one cannot expect in general to have global solutions of a nonlinear differential equations. In fact, the solution of (1.1) is
\[ x(t) = \frac{x_0}{\sqrt[n-1]{1 - (n-1)x_0^{n-1}t}}, \]
so a solution exists in the interval where \((n-1)x_0^{n-1}t < 1\) but it becomes unbounded as \((n-1)x_0^{n-1}t \nearrow 1\).

2. Perturbation of the Cauchy problem

**Lemma 2.1**[2]. Let \(f(t, x)\) and \(\frac{\partial f}{\partial x}(t, x)\) be continuous in an open set \(\Omega \subset \mathbb{R}^{n+1}\). Assume that the Cauchy problem
\[ \begin{cases} \frac{dx(t)}{dt} = f(t, x(t)), \\ x(t_0) = x_0 \end{cases} \]

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has a solution with graph in $\Omega$ for $t_1 \leq t \leq t_2$ where $t_1 < t_0 < t_2$. Then there is a neighborhood $U$ of $x_0$ in $\mathbb{R}^n$ such that for every $y \in U$ the Cauchy problem (2.1) with $x_0$ replaced by $y$ has a unique solution $x(t, y)$ for $t_1 \leq t \leq t_2$. The solution is in $C^1([t_1, t_2] \times U)$.

Take any function $g$ such that $g(t, x)$ and $\frac{\partial g}{\partial x}(t, x)$ are continuous in the open set $\Omega \subset \mathbb{R}^{1+n}$ of the above theorem.

**Corollary 2.2.** Under the conditions of Lemma 2.1 the Cauchy problem

$$
\begin{align*}
\frac{dx(t)}{dt} &= f(t, x(t)) + \epsilon g(t, x(t)), \\
x(t_0) &= y
\end{align*}
$$

has a unique solution $x(t, \epsilon, y)$ for $t_1 \leq t \leq t_2$ where $t_1 < t_0 < t_2$. The solution is in $C^1([t_1, t_2] \times U \times I)$.

**Proof.** Apply the Lemma 2.1 to the following Cauchy problem

$$
\begin{align*}
\frac{dx(t)}{dt} &= f(t, x(t)) + \eta(t)g(t, x(t)), \\
\frac{dx(t)}{dt} &= 0, \\
x(t_0) &= y, \\
\eta(t_0) &= \epsilon.
\end{align*}
$$

The existence and regularity of solution $(x(t), \eta(t))$ of this equation is guaranteed by the theorem. Then $x(t)$ is a solution of the Cauchy problem (2.2). \hfill \square

**Lemma 2.3.** Let $x(t)$ be a solution in $[0, T]$ of the ordinary differential equation

$$
\begin{align*}
\frac{dx(t)}{dt} &= a_0(t)x(t)^n + a_1(t)x(t) + a_2(t), \\
x(0) &= x_0
\end{align*}
$$

with all $a_j, j = 0, 1, 2, \text{ are continuous and } a_0 \geq 0$. Let

$$
K = \int_0^T |a_2(s)| \exp \left( -\int_0^s a_1(u)du \right) ds.
$$

If $x(0) > K$ it follows that

$$
K < \frac{1}{n-1} \left( x(0) - K \right)^{1-n}.
$$
Proof. Set \( x(t) = X(t) \exp \left( \int_0^t a_1(s) ds \right) \). Then
\[
\frac{dx}{dt} = \frac{dX}{dt} \exp \left( \int_0^t a_1(s) ds \right) + a_1(t)X(t) \exp \left( \int_0^t a_1(s) ds \right)
\]
\[
= a_0(t)X(t)^n \exp \left( n \int_0^t a_1(s) ds \right)
\]
\[
+ a_1(t)X(t) \exp \left( \int_0^t a_1(s) ds \right) + a_2(t).
\]
Thus we have
\[
\frac{dX}{dt} = a_0(t) \exp \left( (n-1) \int_0^t a_1(s) ds \right) X(t)^n
\]
\[
+ a_2(t) \exp \left( - \int_0^t a_1(s) ds \right)
\]
\[
= \tilde{a}_0(t)X(t)^n + \tilde{a}_2(t)
\]
where \( \tilde{a}_0(t) = a_0(t) \exp \left( (n-1) \int_0^t a_1(s) ds \right) \) and \( \tilde{a}_2(t) = a_2(t) \exp \left( - \int_0^t a_1(s) ds \right) \). Let’s introduce
\[
X_2(t) = \int_0^t |\tilde{a}_2(s)| ds.
\]
Then \( X_2(0) = 0, X_2(T) = K \). Let \( X_1 \) be the solution of the Cauchy problem
\[
\begin{align*}
\frac{dX_1(t)}{dt} &= \tilde{a}_0(t) \left( X_1(t) - K \right)^n, \\
X_1(0) &= x(0).
\end{align*}
\]
Upon integrating this equation we obtain
\[
\left( X_1(t) - K \right)^{1-n} - \left( x(0) - K \right)^{1-n} = (1-n) \int_0^t \tilde{a}_0(s) ds.
\]
Since \( \tilde{a}_0 \geq 0 \), \( X_1 \) is increasing if \( X_1 \) exists in \([0, T]\) and
\[
\int_0^T \tilde{a}_0(s) ds = \frac{1}{n-1} \left( x(0) - K \right)^{1-n} - \frac{1}{n-1} \left( X_1(T) - K \right)^{1-n}
\]
\[
< \frac{1}{n-1} \left( x(0) - K \right)^{1-n}.
\]
Now
\[
\frac{d}{dt}(X_1(t) - X_2(t)) = \tilde{a}_0(t) \left( X_1(t) - K \right)^n - |\tilde{a}_2(t)| \\
\leq \tilde{a}_0(t) \left( X_1(t) - X_2(t) \right)^n + \tilde{a}_2(t).
\]

Observe that \( X_1(t) - X_2(t) = x(t) \) when \( t = 0 \). Therefore \( X_1(t) - X_2(t) \leq x(t) \) in \([0, t]\) as long as \( X_1(t) \) exists. Thus \( X_1 \) cannot become infinite in \([0, T]\), which proves that
\[
\int_0^T |a_0(t)| dt \cdot \exp \left( (1 - n) \int_0^T |a_1(t)| dt \right) \\
\leq \frac{1}{n-1} \left( x(0) - K \right)^{1-n}.
\]

\[\Box\]

**Theorem 2.4.** Let \( a_j, j = 0, 1, 2, \) be continuous functions in \([0, T]\), set \( a_0^+ = \max(a_0, 0) \), and define \( K \) by (2.4). If \( x_0 \geq 0 \) and
\[
\begin{align*}
\int_0^T a_0^+(t) dt \cdot \exp \left( (n-1) \int_0^T |a_1(t)| dt \right) < (x_0 + K)^{1-n}, \\
\int_0^T |a_0(t)| dt \cdot \exp \left( (n-1) \int_0^T |a_1(t)| dt \right),
\end{align*}
\]
then
\[
\begin{align*}
\frac{dx(t)}{dt} &= a_0(t)x(t)^n + a_1(t)x(t) + a_2(t), \\
x(0) &= x_0
\end{align*}
\]
has a solution in \([0, T]\) with \( x(0) = x_0 \), and
\[
x(T)^{1-n} \geq (x_0 + K)^{1-n} + (1-n) \int_0^T a_0^+(t) dt \cdot \exp \left( (n-1) \int_0^T |a_1(t)| dt \right)
\]
if \( x(T) \geq 0 \).
Proof. We set
\[ x(t) = X(t) \exp \left( \int_0^t a_1(s) \, ds \right). \]

Let \( X_2(t) \) again be the integral of \( |\tilde{a}_2| \) with \( X_2(0) = 0 \). Thus \( X_2(T) = K \). Now let \( X_1 \) be the solution of
\[
\frac{dX_1(t)}{dt} = \tilde{a}_0^+(t) \left( X_1(t) + K \right)^n, \quad X_1(0) = x_0,
\]

which is
\[
\left( X_1(t) + K \right)^{1-n} = (x_0 + K)^{1-n} + (1 - n) \int_0^t \tilde{a}_0^+(s) \, ds.
\]

Since
\[
\int_0^T \tilde{a}_0^+(t) \cdot \exp \left( (n - 1) \int_0^t a_1(s) \, ds \right) \, dt \leq \int_0^T \tilde{a}_0^+(t) \, dt \cdot \exp \left( (n - 1) \int_0^T |a_1(t)| \, dt \right)
\]
we obtain by (2.5) an increasing function \( X_1 \) existing in \([0, T]\). Since
\[
\frac{d}{dt} \left( X_1(t) + X_2(t) \right) = \tilde{a}_0^+(t) \left( X_1(t) + K \right)^n + |\tilde{a}_2(t)| \geq \tilde{a}_0(t) \left( X_1(t) + X_2(t) \right)^n + \tilde{a}_2(t)
\]
and \( X_1 + X_2 = X \) at \( t = 0 \), we have \( X \leq X_1 + X_2 \leq X_1 + K \) in \([0, T]\) if \( X \) exists. Hence
\[
X(T)^{1-n} \geq \left( X_1(T) + K \right)^{1-n}
\]
\[
= (x_0 + K)^{1-n} + (1 - n) \int_0^T \tilde{a}_0^+(s) \, ds
\]
\[
\geq (x_0 + K)^{1-n}
\]
\[
+ (1 - n) \int_0^T \tilde{a}_0^+(t) \, dt \cdot \exp \left( (n - 1) \int_0^T |a_1(t)| \, dt \right).
\]
Since \( x(T)^{1-n} = X(T)^{1-n} \exp \left( (1-n) \int_0^T a_1(t) dt \right) \) we have

\[
x(T)^{1-n} \geq (x_0 + K)^{1-n} + (1-n) \int_0^T a_0^+(t) \exp \left( (n-1) \int_0^T |a_1(t)| dt \right).
\]

(2.6)

When \( a_0 \equiv 1, a_1 \equiv 0, a_2 \equiv 0, \) (2.6) reduces to

\[
T \leq \frac{x_0^{1-n}}{n-1}.
\]

This agrees with (1.2).

References


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