

NOTE ON T-PARTS OF BCI-ALGEBRAS RELATIVE TO SUBSETS

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ABSTRACT. In this paper, we introduce the notion of T -part of BCI-algebras relative to a subset. We show that if A is a subalgebra of a BCI-algebra X , then so is the T -part $T_A(X)$ of X relative to A . We provide equivalent conditions that the T -part of X relative to A is an ideal.

1. Introduction

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki [I] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Recently, X. Zhang and Y. B. Jun [ZJ] introduced the notion of T -part in a BCI-algebra X and showed that the T -part of X is a subalgebra of X .

In this paper, we introduce the notion of T -part of BCI-algebras relative to subsets. We give equivalent conditions that the T -part of X relative to A is an ideal.

We begin with some definitions and properties that will be useful in our results.

DEFINITION 1.1. A BCI-algebra is an algebra $(X; *, 0)$ of type $(2, 0)$ is satisfying the following axioms for all $x, y, z \in X$:

- (a) $((x * y) * (x * z)) * (z * y) = 0$
- (b) $(x * (x * y)) * y = 0$
- (c) $x * x = 0$
- (d) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For any BCI-algebra X , the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on X .

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A BCI-algebra X has the following properties for all $x, y, z \in X$:

- (1) $x * 0 = x$
- (2) $(x * y) * z = (x * z) * y$
- (3) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

DEFINITION 1.2. A non-empty subset I of a BCI-algebra X is called an *ideal* of X if it satisfies

- (I1) $0 \in I$,
- (I2) $x * y \in I$ and $y \in I$ imply $x \in I$.

It is well known that an ideal I of a BCI-algebra X need not be a subalgebra.

DEFINITION 1.3. Let X be a BCI-algebra and let A be a non-empty subset of X . The set

$$T_A(X) = \{(0 * x) * x \mid x \in A\}$$

is called the *T-part of X relative to A*. If $A = X$, then $T_X(X) := T(X)$ is called the *T-part of X* ([ZJ]).

Clearly, if $0 \in A$, then $0 \in T_A(X)$.

EXAMPLE 1.4. If X is an associative BCI-algebra or a BCK-algebra, then $T_A(X) = \{0\}$, for every nonempty subset A of X . If X is a BCI-algebra, then $T_{G(X)}(X) = \{0\}$, where $G(X) = \{x \in X \mid 0 * x = x\}$.

DEFINITION 1.5. [MX] An element a of X is called an *atom* if, for all $x \in X$, $x * a = 0$ implies $x = a$. The set $L(X) := \{a \in X \mid a \text{ is an atom of } X\}$ is called the *p-semisimple part* of X . For any $a \in L(X)$, the set $V(a) := \{x \in X \mid a \leq x\}$ is called a *branch* of X . The set $X_+ := \{x \in X \mid 0 \leq x\}$ is called the *BCK-part* of X .

Clearly, 0 is an atom of X . We know that $(L(X), *, 0)$ is a *p-semisimple BCI-algebra* and $(X_+, *, 0)$ is a *BCK-algebra*. It is clear that $V(0) = X_+$ ([MX]).

2. T-parts relative to subsets

PROPOSITION 2.1. If A is a subalgebra of a BCI-algebra X , then so is $T_A(X)$.

Proof. Let $x, y \in T_A(X)$. Then $x = (0 * a) * a$ and $y = (0 * b) * b$ for some $a, b \in A$. Hence

$$\begin{aligned}
 x * y &= ((0 * a) * a) * ((0 * b) * b) \\
 &= ((0 * ((0 * b) * b)) * a) * a \\
 &= (((0 * (0 * b)) * (0 * b)) * a) * a \\
 &= (((0 * a) * (0 * b)) * a) * a \\
 &= ((0 * (a * b)) * (0 * b)) * a \\
 &= ((0 * (0 * b)) * (a * b)) * a \\
 &= ((0 * a) * (0 * b)) * (a * b) \\
 &= (0 * (a * b)) * (a * b).
 \end{aligned}$$

Thus $x * y \in T_A(X)$. This completes the proof. $\square$

COROLLARY 2.2. ([ZJ]) *If X is a BCI-algebra, then $T(X)$ is a sub-algebra of X .*

PROPOSITION 2.3. *If $A \subset B$, then $T_A(X) \subset T_B(X)$.*

Proof. Let $x \in T_A(X)$. Then $x = (0 * a) * a$ for some $a \in A$. Since $A \subset B$, $x \in T_B(X)$. Thus, $T_A(X) \subset T_B(X)$. $\square$

COROLLARY 2.4. *If A is a non-empty subset of X , then $T_A(X) \subset T(X)$.*

PROPOSITION 2.5. *If $f : X \rightarrow Y$ is a surjective homomorphism from a BCI-algebra X onto a BCI-algebra Y , then $f(T_A(X)) = T_{f(A)}(Y)$.*

Proof. Let $x \in T_{f(A)}(Y)$. Then $x = (0 * y) * y$ for some $y \in f(A)$. Since $y = f(a)$, there exists $a \in A$ such that $y = f(a)$. It follows that $x = (0 * f(a)) * f(a) = f((0 * a) * a)$ since f is a homomorphism. Thus, $x \in f(T_A(X))$. Conversely, let $x \in f(T_A(X))$. Then $x = f(a)$ for some $a \in T_A(X)$. Hence there exists $b \in A$ such that $a = (0 * b) * b$. Thus $x = f(a) = f((0 * b) * b) = (0 * f(b)) * f(b)$. Since $f(b) \in f(A)$, we have $x \in T_{f(A)}(Y)$. This completes the proof. $\square$

COROLLARY 2.6. *If $f : X \rightarrow Y$ is a surjective homomorphism from a BCI-algebra X onto a BCI-algebra Y , then $f(T(X)) = T(Y)$.*

THEOREM 2.7. *Let X be a BCI-algebra and let A be a non-empty subset of X . The following are equivalent :*

- (i) $T_A(X)$ is an ideal of X ,
- (ii) $x * a = y * a$ implies $x = y$ for all $x, y \in X_+$ and $a \in T_A(X)$,
- (iii) $x * a = 0 * a$ implies $x = 0$ for all $x \in X_+$ and $a \in T_A(X)$.

Proof. (i) $\Rightarrow$ (ii) Assume that $T_A(X)$ is an ideal of X and let $x * a = y * a$ for all $x, y \in X_+$ and $a \in T_A(X)$. Then $(x * y) * a = (x * a) * y = (y * a) * y = (y * y) * a = 0 * a \in T_A(X)$. Since $T_A(X)$ is an ideal of X , we have $x * y \in T_A(X)$, we have $x * y \in T_A(X)$. On the other hand, note that $x * y \in X_+$ and $X_+ \cap T_A(X) \subseteq X_+ \cap L(X) = \{0\}$. It follows that $x * y = 0$ and hence, $x \leq y$. Similarly, we get $y \leq x$, and therefore $x = y$.

(ii) $\Rightarrow$ (iii) It is straightforward.

(iii) $\Rightarrow$ (i) Suppose that (iii) holds. Assume that $x * y \in T_A(X)$ and $y \in T_A(X)$ for all $x, y \in X$. Denote $t = 0 * (0 * x)$. Then $t \in L(X)$. Since $t = 0 * (0 * x) \leq x$, it follows from (3) that $t * y \leq x * y$. Hence $x * y \in V(t * y)$. From $x * y \in T_A(X) \subseteq L(X)$, we have $x * y = t * y$. Thus $(x * t) * y = (x * y) * t = (t * y) * t = (t * t) * y = 0 * y$. It follows from (iii) that $x * (0 * (0 * x)) = x * t = 0$, since $x * t \in X_+$. Thus $x = 0 * (0 * x) \in L(X)$. Since $T_A(X)$ is an ideal of $L(X)$, we have $x \in T_A(X)$, and hence $T_A(X)$ is an ideal of X . This completes the proof. $\square$

DEFINITION 2.8. ([Z]) A non-empty subset A of a BCI-algebra X is called a T -ideal of X if it satisfies :

- (T1) $0 \in A$,
- (T2) $x * (y * z) \in A$ and $y \in A$ imply $x * z \in A$ for all $x, y, z \in X$.

It is well known that every T -ideal of a BCI-algebra is an ideal (see, [Z, Theorem 1]), but not conversely ([ZJ]).

LEMMA 2.9. ([Z]) *Let A be a T -ideal of a BCI-algebra X . Then $(0 * x) * x \in A$ for all $x \in X$.*

PROPOSITION 2.10. *If A is a T -ideal of a BCI-algebra X , then we have $T_A(X) \subseteq A$.*

Proof. If $x \in T_A(X)$, then $x = (0 * a) * a$ for some $a \in A$. By Lemma 2.9, we have $x = (0 * a) * a \in A$. This completes the proof. $\square$

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