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CHARACTERIZATION OF OPERATORS TAKING COMPACT SETS INTO SUBSETS OF THE RANGES OF VECTOR MEASURES

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Abstract. We characterize operators between Banach spaces sending compact sets into sets that lie in the range of a vector measure. Further, we describe operators between Banach spaces taking compact sets into sets that lie in the range of a vector measure of bounded variation.

1. Introduction

The intriguing connection between the geometry of subsets of Banach spaces and vector measure theory is not confined to Radon-Nikodym considerations. Questions regarding the finer structure of the range of a vector measure have found interest since Liapounoff's discovery of his everintriguing convexity theorem which states that the range of a nonatomic vector measure with values in a finite dimensional space is compact and convex. The infinite dimensional version of Liapounoff's theorem remained resistant to analysis for a long time. It is an important fact, first established by Bartle, Dunford and Schwartz in the early fifties, that the range of a vector measure is always relatively weakly compact.

Among the relatively weakly compact subsets of Banach spaces, those that are the range of a vector measure occupy a special place ; a remarkable similarity to the relatively norm compact sets is evidenced. For instance, Diestel and Seifert [3] proved that any sequence in the range of a vector measure admits a subsequence with norm convergent arithmetic means, a phenomenon not shared by all weakly compact sets.

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Any intuition gained by noting the similarities between relatively norm compact sets and sets arising as ranges of vector measures must be tempered by the fact that the closed unit ball of an infinite dimensional Banach space can be the range of a vector measure.

Anantharaman and Garg [1] proved that the closed unit ball of a Banach space X is the range of a vector measure if and only if the dual of a Banach space X is isometrically isomorphic to a reflexive subspace of $L^1(\mu)$ for some probability measure μ.

Anantharaman and Diestel [2] found that every weakly compact subset of $BD1$ (the separable $\mathcal{L}_\infty$ space of Bourgain and Delbaen that has the weakly compact extension property) lies inside the range of a $BD1$-valued measure. They also gave some necessary and some sufficient conditions for a sequence in a Banach space X to lie in the range of an X-valued measure.

Piñeiro and Rodriguez-Piazza [11] showed that the compact subset of a Banach space X lies inside the range of an X-valued measure if and only if the dual of a Banach space X can be embedded into an $L^1(\mu)$-space for a suitable measure μ.

It is an easy consequence of the celebrated Dvoretsky-Rogers theorem that given an infinite dimensional Banach space X, there is an X-valued measure that does not have finite variation [14]. Thus the question arose : Which Banach spaces X have the property that every compact subset of X lies inside the range of an X-valued measure of bounded variation ? This was answered by Piñeiro and Rodriguez-Piazza [11].

In this paper we deal with the above mentioned problems in the framework of operators acting between Banach spaces. Here, we present Piñeiro's approach to this subject [12].

Piñeiro introduced the space $\mathcal{R}(X, Y)$ of all operators from a Banach space X into a Banach space Y taking compact subsets of X into subsets that lie in the range of a Y-valued measure. In addition, he defined $\mathcal{R}_{bv}(X, Y)$ as the set of all operators from a Banach space X into a Banach space Y sending compact subsets of X into subsets that lie in the range of a Y-valued measure with bounded variation.

We first give usable necessary and sufficient conditions for an operator to belong to the space $\mathcal{R}$.

Next we provide a description of operators belonging to the space

$\mathcal{R}_{bv}$ in terms of $(1,\infty,1)$-summing operators.

Finally we see how the space $\mathcal{R}$ is linked with the space $\mathcal{R}_{bv}$.

2. Definitions and Notation

We present some of the definitions and notation to be used. Throughout this paper X and Y denote Banach spaces.

A function μ from a σ-field Σ of subsets of a set Ω to a Banach space X is called a countably additive vector measure if $\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ in the norm topology of X for all sequences (E_n) of pairwise disjoint members of Σ such that $\cup_{n=1}^{\infty} E_n \in \Sigma$. The range of μ will be denoted by $\operatorname{rg}\mu$. The variation of μ is the extended nonnegative function $|\mu|$ whose value on a set $E \in \Sigma$ is given by $|\mu|(E) = \sup_{\pi} \sum_{A\in\pi} \|\mu(A)\|$, where the supremum is taken over all partitions π of E into a finite number of pairwise disjoint members of Σ. If $|\mu|(\Omega) = \operatorname{tv}(\mu) < \infty$ then μ will be called a measure of bounded variation. The semivariation of μ is the extended nonnegative function $\|\mu\|$ whose value on a set $E \in \Sigma$ is given by $\|\mu\|(E) = \sup\{|x^* \circ \mu|(E) : x^* \in X^*, \|x^*\| \leq 1\}$, where $|x^* \circ \mu|$ is the variation of the real-valued measure $x^* \circ \mu$. If $\|\mu\|(\Omega) = \operatorname{tsv}(\mu) < \infty$, then μ will be called a measure of bounded semivariation.

Notation. (1) The dual of a Banach space X is denoted by X^*.
(2) The closed unit ball of a Banach space X is denoted by B_X.
(3) The unit sphere of a Banach space X is denoted by S_X.
(4) The dual operator of an operator T is denoted by T^*.
(5) $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y.
(6) The canonical isometric embedding from a Banach space Y into the bidual of Y is denoted by κ_Y.

The space $\mathcal{R}(X)$ is defined to consist of all sequences (x_n) in X such that there exists an X-valued measure μ satisfying $\{x_n : n \in \mathbb{N}\} \subset \operatorname{rg}\mu$. For each $(x_n) \in \mathcal{R}(X)$, define $\|(x_n)\|_r = \inf \operatorname{tsv}(\mu)$, where the infimum is taken over all vector measures μ as above.

The space $\mathcal{R}_c(X)$ consists of all sequences in X that lie inside the range of an X-valued measure with relatively compact range. If (x_n)

belongs to $\mathcal{R}_c(X)$ then proposition 1.4. of [11] ensures that there exists an unconditionally convergent series $\sum_{k=1}^{\infty} y_k$ in X for which $\{x_n : n \in \mathbb{N}\} \subset \{\sum_{k=1}^{\infty} \alpha_k y_k : (\alpha_k) \in \ell_\infty, \|(\alpha_k)\|_\infty \leq 1\}$. For each $(x_n) \in \mathcal{R}_c(X)$, define $\|(x_n)\|_{rc} = \inf \sup\{\sum_{k=1}^{\infty} |\langle x^*, y_k\rangle| : x^* \in B_{X^*}\}$, where the infimum is taken over all unconditionally convergent series $\sum_{k=1}^{\infty} y_k$ of the kind described above.

The space $\mathcal{R}_{bv}(X)$ is defined to consist of all sequences (x_n) in X such that there exists an X-valued measure μ with bounded variation satisfying $\{x_n : n \in \mathbb{N}\} \subset \operatorname{rg} \mu$. For each $(x_n) \in \mathcal{R}_{bv}(X)$, set $\|(x_n)\|_{bv} = \inf \operatorname{tv}(\mu)$, where the infimum is taken over all vector measures μ as above.

The space $\mathcal{R}_{bvc}(X)$ consists of all sequences (x_n) in X such that there exists an absolutely convergent series $\sum_{k=1}^{\infty} y_k$ in X satisfying $\{x_n : n \in \mathbb{N}\} \subset \{\sum_{k=1}^{\infty} \alpha_k y_k : (\alpha_k) \in \ell_\infty, \|(\alpha_k)\|_\infty \leq 1\}$. For each $(x_n) \in \mathcal{R}_{bvc}(X)$, let $\|(x_n)\|_{bvc} = \inf \sum_{k=1}^{\infty} \|y_k\|$, where the infimum is extended over all such absolutely convergent series $\sum_{k=1}^{\infty} y_k$.

We denote by $C_0(X)$ the space of all sequences (x_n) in X with $\lim_{n\to\infty} \|x_n\| = 0$.

We write $\mathcal{R}(X, Y)$ (respectively $\mathcal{R}_c(X, Y)$) for the set of all operators T from X into Y such that for each sequence $(x_n) \in C_0(X)$, the sequence (Tx_n) belongs to $\mathcal{R}(Y)$ (respectively $\mathcal{R}_c(Y)$).

We denote by $\mathcal{R}_{bv}(X, Y)$ (respectively $\mathcal{R}_{bvc}(X, Y)$) the space of all operators T from X into Y such that for each sequences $(x_n) \in C_0(X)$, the sequence (Tx_n) belongs to $\mathcal{R}_{bv}(Y)$ (respectively $\mathcal{R}_{bvc}(Y)$).

Let X_0^N (respectively Y_0^N) denote the linear space of all vector sequences (x_n) in X (respectively Y) such that the set of n for which $x_n \neq 0$ is finite.

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. We say that the operator $T : X \to Y$ belongs to $\mathcal{A}^*(X, Y)$ provided there is a constant $C \geq 0$ such that regardless of the finite dimensional normed spaces E and F and operators $a \in \mathcal{L}(E, X), b \in \mathcal{L}(Y, F)$ and $U \in \mathcal{L}(F, E)$, the composition $E \xrightarrow{a} X \xrightarrow{T} Y \xrightarrow{b} F \xrightarrow{U} E$ satisfies $|\operatorname{tr}(UbTa)| \leq C \cdot \|a\| \cdot \|b\| \cdot \alpha(U)$. The collection of all such C has an infimum, which is denoted by $\alpha^*(T)$. The Banach operator ideal $[\mathcal{A}^*, \alpha^*]$ is called the adjoint operator ideal of $[\mathcal{A}, \alpha]$.

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. We introduce the notation $\mathcal{A}^d(X, Y)$ for the set of all $T \in \mathcal{L}(X, Y)$ with $T^* \in \mathcal{A}(Y^*, X^*)$. For such T's we also stipulate that $\alpha^d(T) = \alpha(T^*)$. A Banach operator ideal $[\mathcal{A}^d, \alpha^d]$ is called the dual operator ideal of $[\mathcal{A}, \alpha]$.

For $1 \leq p \leq \infty$, an operator $T \in \mathcal{L}(X, Y)$ is called p-integral if there are a probability measure μ and operators $A \in \mathcal{L}(L^p(\mu), Y^{**})$ and $B \in \mathcal{L}(X, L^\infty(\mu))$ such that $\kappa_Y \circ T = A \circ i_p \circ B$, where $i_p : L^\infty(\mu) \to L^p(\mu)$ is the formal identity. The p-integral norm of T is defined by $\imath_p(T) = \inf\{\|A\|\|B\|\}$, where the infimum is extended over all measures μ and operators A and B as above. The collection of all p-integral operators from X into Y is denoted by $\mathcal{I}_p(X, Y)$.

Let $1 \leq p < \infty$. The vector sequence (x_n) in X is weakly p-summable if the scalar sequences $(\langle x^*, x_n \rangle)$ are in ℓ_p for every $x^* \in X^*$. We denote by $\ell_p^{\text{weak}}(X)$ the set of all such sequences in X. This is a Banach space under the norm

$$\|(x_n)\|_p^{\text{weak}} = \sup\big\{(\sum_n |\langle x^*, x_n \rangle|^p)^{1/p} : x^* \in X^*,\ \|x^*\| \leq 1\big\}.$$

An operator $T \in \mathcal{L}(X, Y)$ is said to be nuclear if it can be written in the form $T = \sum_{i=1}^\infty \tau_i \, x_i^* \bigotimes y_i$ with (x_i^*) in B_{X^*}, (y_i) in B_Y and $(\tau_i) \in \ell_1$. The set of these operators is denoted by $\mathcal{N}(X, Y)$. For $T \in \mathcal{N}(X, Y)$ we define $\nu(T) = \inf \sum_{i=1}^\infty |\tau_i|$, with the infimum taken over all nuclear representations of T as above.

For $1 \leq p < \infty$, an operator $T \in \mathcal{L}(X, Y)$ is called absolutely p-summing if there exists a constant $C \geq 0$ such that for any finite subset $\{x_i\}_{i=1}^n \subset X$, we have

$$(\sum_{i=1}^n \|Tx_i\|^p)^{1/p} \leq C \cdot \sup\big\{(\sum_{i=1}^n |\langle x^*, x_n \rangle|^p)^{1/p} : x^* \in X^*,\ \|x^*\| \leq 1\big\}.$$

The infimum of such C is the absolutely p-summing norm of T and denoted by $\pi_p(T)$. We write $\Pi_p(X, Y)$ for the set of all absolutely p-summing operators from X into Y.

Let $1 \leq q < \infty$, $1 \leq p, r \leq \infty$ and $1/q \leq 1/p + 1/r$. An operator $T \in \mathcal{L}(X, Y)$ is called absolutely (q, p, r)-summing if there exists a constant $C \geq 0$ such that for all finite subset $\{x_i\}_{i=1}^n \subset X$ and $\{y_i^*\}_{i=1}^n \subset Y^*$, we have

$$(\sum_{i=1}^n |\langle Tx_i, y_i^* \rangle|^q)^{1/q} \leq C \cdot \|(x_i)\|_p^{\text{weak}} \cdot \|(y_i^*)\|_r^{\text{weak}}.$$

The infimum of such C is the absolutely (q,p,r)-summing norm of T and denoted by $\pi_{q,p,r}(T)$. We write $\Pi_{q,p,r}(X,Y)$ for the set of all absolutely (q,p,r)-summing operators from X into Y.

3. Results

Let us start with the problem which gives a description of operators belonging to the space $\mathcal{R}$ in terms of ∞-integal operators.

THEOREM 1. *Let $T \in \mathcal{L}(X,Y)$. Then the following statements are equivalent.*

(i) $T \in \mathcal{R}(X,Y)$

(ii) $T \in \mathcal{R}_c(X,Y)$

(iii) *Regardless of the Banach space Z and the operator $U \in \mathcal{I}_\infty^d(Z,X)$, the composition $TU : Z \to Y$ is ∞-integral.*

(iv) *There is a constant $C > 0$ for which $\sum_{i=1}^m \|T^* u_i^*\| \le C \sum_{j=1}^n \|v_j^*\|$ whenever $(u_i^*)_{i=1}^m$ and $(v_j^*)_{j=1}^n$ are finite sequences in Y^* satisfying $\sum_{i=1}^m |\langle u_i^*, y\rangle| \le \sum_{j=1}^n |\langle v_j^*, y\rangle|$ for all $y \in Y$.*

Proof. (i)$\Rightarrow$(ii). Assume that $T \in \mathcal{R}(X,Y)$. Then for every finite subset $E = \{x_1, \cdots, x_n\}$ of B_X there is a Y-valued measure μ with bounded semivariation satisfying $T(E) \subset \mathrm{rg}\,\mu$ and $\mathrm{tsv}(\mu) \le C$ for some constant $C > 0$. Choose $A_1, \cdots, A_n \in \Sigma$ such that $\mu(A_i) = Tx_i$ for $1 \le i \le n$ and consider the field Σ_0 generated by $\{A_1, \cdots, A_n\}$. If $E_1, \cdots, E_m \in \Sigma_0$ are the atoms, we write $\mu(E_j) = y_j$ for $1 \le j \le m$. This assures us that $T(E) \subset \{\sum_{j=1}^m \alpha_j y_j : \|(\alpha_j)\|_\infty \le 1\}$ and $\sup\{\sum_{j=1}^m |\langle y^*, y_j\rangle| : y^* \in B_{Y^*}\} \le \mathrm{tsv}(\mu) \le C$. Now we define a linear map $\hat{T} : X_0^N \to Y_0^N$ by $\hat{T}(x_n) = (Tx_n)$ for all $(x_n) \in X_0^N$, where X_0^N (respectively Y_0^N) is equipped with the norm $\|\cdot\|_\infty$ (respectively $\|\cdot\|_{rc}$). Then $\|\hat{T}(x_n)\|_{rc} = \|(Tx_n)\|_{rc} \le C' \|(x_n)\|_\infty$ for all $(x_n) \in X_0^N$ and hence $\hat{T}$ is bounded. Since X_0^N is dense in $C_0(X)$, it follows that $(Tx_n) \in \mathcal{R}_c(Y)$ for all $(x_n) \in C_0(X)$ with $\|(Tx_n)\|_{rc} \le C' \|(x_n)\|_\infty$. This yields that $T \in \mathcal{R}_c(X,Y)$.

(ii)$\Rightarrow$(iii). Let F be a finite dimensional subspace of Y^*. Given $\epsilon > 0$ we select $x_1, \cdots, x_n$ in S_X so that $\|x^*\| \le (1+\epsilon)\sup\{|\langle x^*, x_i\rangle| : i = 1, \cdots, n\}$ for all $x^* \in T^*(F)$. $(\cdot)$

The hypothesis (ii) tells us that $(Tx_i)_{i=1}^n \in \mathcal{R}_c(Y)$. Proposition 1.4. of [11] guarantees the existence of an unconditionally convergent series $\sum y_k$ in Y such that $\{Tx_i : 1 \le i \le n\} \subset \{\sum \alpha_k y_k : \|(\alpha_k)\|_\infty \le 1\}$ and $\sup\{\sum |\langle y^*, y_k\rangle| : y^* \in B_{Y^*}\} \le C$ for some constant $C > 0$. We use this series to define an operator $S : F \to \ell_1$ via $Sy^* = (\langle y^*, y_k\rangle)$ for all $y^* \in F$. It is obvious that $\|S\| \le C$. Note that if $Sy^* = 0$ for all $y^* \in F$ then $|\langle T^*y^*, x_i\rangle| = |\langle y^*, Tx_i\rangle| \le \sum |\langle y^*, y_k\rangle| = 0$ for $1 \le i \le n$ and thus we can call on $(\cdot)$ to obtain $T^*y^* = 0$ for all $y^* \in F$. This permits us to well-define an operator $R : S(F) \to X^*$ by $R(Sy^*) = T^*y^*$ for all $y^* \in F$. By another use of $(\cdot)$ we have $\|R\| \le 1 + \epsilon$. The construction of S and R informs us that $T^*|_F$ is factorizable through a subspace of ℓ_1. We apply a result due to Lindenstrauss and Pelczynski [9] to derive that T^* admits a factorization through a subspace of an $L^1(\mu)$-space for some measure μ. Kwapień's theorem [8] steps in to ensure that statement (iii) holds.

(iii)⇒(i). The hypothesis (iii) allows us to use Kwapień's theorem [8] to get that T^* factors through a subspace of an $L^1(\lambda)$-space for a suitable measure λ. Then T^{**} can be factored through a quotient of an $L^\infty(\lambda)$-space for some measure λ as follows : $T^{**} : X^{**} \xrightarrow{B} L^\infty(\lambda)/N \xrightarrow{A} Y^{**}$. First we will show that $C_0(L^\infty(\lambda)/N) \subset R(L^\infty(\lambda)/N)$. $(\cdot)$

To this end, we take $\bar{f}_1, \cdots, \bar{f}_n$ in $B_{(L^\infty(\lambda)/N)}$. Given $\epsilon > 0$ we can pick $f_i \in \bar{f}_i$ so that $\|f_i\| \le 1 + \epsilon$ for $1 \le i \le n$. Since $L^\infty(\lambda)$ is an $\mathcal{L}_{\infty,1+\epsilon}$-space, $\{f_1, \cdots, f_n\}$ sits inside a finite dimensional subspace E of $L^\infty(\lambda)$ for which we can find an isomorphism $v : \ell_\infty^m \to E$, where $m = \dim E$, with $\|v^{-1}\| = 1$ and $\|v\| \le 1 + \epsilon$. Since v is weakly compact, we invoke proposition 1.3 of [11] to infer that there exists an E-valued measure μ such that $\{f_1, \cdots, f_n\} \subset v(B_{\ell_\infty^m}) = \operatorname{rg}\mu$ and $\operatorname{tsv}(\mu) \le 2(1 + \epsilon)$. Then $\mu_1 = q_N \circ \mu$, where $q_N : L^\infty(\lambda) \to L^\infty(\lambda)/N$ is the natural quotient map, is an $L^\infty(\lambda)/N$-valued measure so that $\{\bar{f}_1, \cdots, \bar{f}_n\} \subset \operatorname{rg}\mu_1$ and $\operatorname{tsv}(\mu_1) \le 2(1 + \epsilon)$, which verifies $(\cdot)$.

Now let us take $(x_n) \in C_0(X)$. Then $(x_n) \in C_0(X^{**})$ and hence $(Bx_n) \in C_0(L^\infty(\lambda)/N)$. From $(\cdot)$ we know that $(Bx_n) \in R(L^\infty(\lambda)/N)$ and so $(T^{**}x_n) = (ABx_n) \in R(Y^{**})$. This gives $\kappa_Y T \in \mathcal{R}(X, Y^{**})$. As a result for every finite subset $E = \{x_1, \cdots, x_n\}$ of B_X there exists a Y^{**}-valued measure ν with bounded semivariation satisfying $\kappa_Y T(E) \subset \operatorname{rg}\nu$ and $\operatorname{tsv}(\nu) \le C$ for some constant $C > 0$. Select $A_1, \cdots, A_n \in \Sigma$ with $\nu(A_i) = \kappa_Y Tx_i$ for $1 \le i \le n$ and consider the

field Σ_0 generated by $\{A_1, \cdots, A_n\}$. Let $\tilde{\nu}$ be the restriction of ν to Σ_0 and let F be the finite dimensional linear span of $\mathrm{rg}(\tilde{\nu})$. Thanks to the principle of local reflexivity, for each $\epsilon > 0$, there is an injective operator $u : F \to Y$ such that $uy = y$ for all $y \in F \cap Y$ and $\|u\| \leq 1+\epsilon$. Therefore $\nu_1 = u \circ \tilde{\nu}$ is a Y-valued measure for which $T(E) \subset \mathrm{rg}\, \nu_1$ and $\mathrm{tsv}(\nu_1) \leq (1+\epsilon)\, C$. This forces $T \in \mathcal{R}(X, Y)$.
(iii)⇒(iv). On account of hypothesis (iii), we take account of Kwapień's theorem [8] to deduce that there exist a subspace L of a space $L^1(\mu)$ and a factorization $T^* : Y^* \xrightarrow{W} L \xrightarrow{V} X^*$ with $\|v\| \cdot \|w\| \leq C$. Let $(u_i^*)_{i=1}^m$ and $(v_j^*)_{j=1}^n$ be finite sequences in Y^* satisfying

$$(\cdot) \qquad \sum_{i=1}^m |\langle u_i^*, y\rangle| \leq \sum_{j=1}^n |\langle v_j^*, y\rangle| \text{ for all } y \in Y.$$

We use these sequences to define operators $R : Y \to \ell_1^m$ and $S : Y \to \ell_1^n$ via $Ry = (\langle u_i^*, y\rangle)_{i=1}^m$ for all $y \in Y$ and $Sy = (\langle v_j^*, y\rangle)_{j=1}^n$ for all $y \in Y$, respectively. Our condition $(\cdot)$ translates to read $\|Ry\| \leq \|Sy\|$ for all $y \in Y$, and this allows us to well-define an operator $A : S(Y) \to \ell_1^n$ by $A(Sy) = Ry$ for all $y \in Y$. Plainly, $\|A\| \leq 1$. Thinking of W as an operator with values in $L^1(\mu)$, we can apply Kwapień's result [7] to produce

$$\begin{aligned}\sum_{i=1}^m \|T^* u_i^*\| &= \sum_{i=1}^m \|VWu_i^*\| \leq \|V\| \sum_{i=1}^m \|WR^* e_i\| \leq \|V\|\, \pi_1(WR^*) \\ &= \|V\|\, \pi_1(RW^*) = \|V\|\, \pi_1(ASW^*) \leq \|V\|\, \|W\|\, \pi_1(S) \leq C\, \pi_1(S).\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}\pi_1(S) &= \sup\{\sum_{k=1}^\infty \|Sy_k\| : \|(y_k)\|_1^{\mathrm{weak}} \leq 1\} \\ &= \sup\{\sum_{k=1}^\infty \sum_{j=1}^n |\langle S^* e_j, y_k\rangle| : \|(y_k)\|_1^{\mathrm{weak}} \leq 1\} \\ &\leq \sum_{j=1}^n \|S^* e_j\| = \sum_{j=1}^n \|v_j^*\|.\end{aligned}$$

Accordingly we end up with $\sum_{i=1}^m \|T^*u_i^*\| \le C \sum_{j=1}^n \|v_j^*\|$.
(iv)⇒(iii). Let F be a finite dimensional subspace of Y^*, and write K for the norm compact unit sphere of F^*. Denote by Φ the collection of all functions $\varphi : K \to \mathbb{R}$ which are defined by $\varphi(y) = \sum_{i=1}^m |\langle u_i^*, y\rangle| - \sum_{j=1}^n |\langle v_j^*, y\rangle|$, $y \in K$, where $(u_i^*)_{i=1}^m$ and $(v_j^*)_{j=1}^n$ are finite sequences in F such that $\sum_{i=1}^m \|T^*u_i^*\| > C \sum_{j=1}^n \|v_j^*\|$. The hypothesis (iv) leads us to have that $\sup\{\varphi(y) : y \in K\} > 0$ for each $\varphi \in \Phi$. Let $\Psi = \{f \in C(K, \mathbb{R}) : f(y) < 0 \text{ for all } y \in K\}$. Then the separation theorem and the Riesz representation theorem furnish us with a measure μ in $C(K, \mathbb{R})^*$ and a real number α such that $\langle \mu, f\rangle < \alpha \le \langle \mu, \varphi\rangle$ for all $f \in \Psi$ and $\varphi \in \Phi$. Moreover, the properties of Ψ and Φ entail that $\alpha = 0$, so that μ is a non-trivial posive measure. Our choice of K guarantees that $0 < \sup_{y^* \in B_F} \int_K |\langle y, y^*\rangle| d\mu(y) \le \mu(K)$, and this enables us to scale μ so that $C = \sup_{y^* \in B_F} \int_K |\langle y, y^*\rangle| d\mu(y)$. We consider the operator $W : F \to L^1(\mu)$ which is given by $W(y^*) = \langle y^*, \cdot\rangle$ for all $y^* \in F$. Now we assume that $\|T^*u^*\| > C$ for $u^* \in F$ and $v^* \in B_F$. Then the singletons $\{u^*\}$ and $\{v^*\}$ give rise to an element φ of Φ with $\varphi(y) = |\langle u^*, y\rangle| - |\langle v^*, y\rangle|$ for all $y \in K$. As $\langle \mu, \varphi\rangle \ge 0$, we obtain $\|Wu^*\| = \int_K |\langle u^*, y\rangle| d\mu(y) \ge \int_K |\langle v^*, y\rangle| d\mu(y)$, and a passage to the supremum over all $v^* \in B_F$ leads to $\|Wu^*\| \ge C$. In other words, we arrive at the conclusion that $\|T^*u^*\| \le \|Wu^*\|$ for all $u^* \in F$. This signifies that there exists an operator $V : W(F) \to X^*$, of norm at most one, such that $V(Wy^*) = T^*y^*$ for all $y^* \in F$. Consequently $T^*|_F$ is factorizable through a subspace of $L^1(\mu)$. An appeal to a result of Lindenstrauss and Pelczynski [9] establishes that T^* admits a factorization through a subspace of $L^1(\mu)$. Kwapień's theorem [8] assures us that statement (iii) holds. $\square$

In the next theorem we characterize operators belonging to the space $\mathcal{R}_{bv}$ in terms of $(1, \infty, 1)$-summing operators.

Theorem 2. *The following statements about an operator $T : X \to Y$ are equivalent.*

(i) $T \in \mathcal{R}_{bv}(X, Y)$
(ii) $T \in \mathcal{R}_{bvc}(X, Y)$
(iii) $T \in \Pi_{1,\infty,1}(X, Y)$

Proof. (i)⇒(ii). The hypothesis (i) tells us that for every finite subset $E = \{x_1, \cdots, x_n\}$ of B_X there exists a Y-valued measure μ with bounded variation satisfying $T(E) \subset \operatorname{rg}\mu$ and $\operatorname{tv}(\mu) \le C$ for some constant $C > 0$. Choose $A_1, \cdots, A_n \in \Sigma$ such that $\mu(A_i) = Tx_i$ for $1 \le i \le n$ and let Σ_0 be the field generated by $\{A_1, \cdots, A_n\}$. If $E_1, \cdots, E_m \in \Sigma_0$ are the atoms, we set $\mu(E_j) = y_j$ for $1 \le j \le m$. This indicates that $T(E) \subset \{\sum_{j=1}^m \alpha_j y_j : \|(\alpha_j)\|_\infty \le 1\}$ and $\sum_{j=1}^m \|y_j\| \le |\mu|(\cup_{j=1}^m E_j) \le \operatorname{tv}(\mu) \le C$. Now we consider a linear map $\hat{T} : X_0^N \to Y_0^N$ which is given by $\hat{T}(x_n) = (Tx_n)$ for all $(x_n) \in X_0^N$, where X_0^N (respectively Y_0^N)is equipped with the norm $\|\cdot\|_\infty$ (respectively $\|\cdot\|_{bvc}$). Then $\|\hat{T}(x_n)\|_{bvc} = \|(Tx_n)\|_{bvc} \le C' \|(x_n)\|_\infty$ for all $(x_n) \in X_0^N$ and so $\hat{T}$ is bounded. We deduce from the density of X_0^N in $C_0(X)$ that $(Tx_n) \in \mathcal{R}_{bvc}(Y)$ for all $(x_n) \in C_0(X)$ with $\|(Tx_n)\|_{bvc} \le C' \|(x_n)\|_\infty$. This implies that $T \in \mathcal{R}_{bvc}(X, Y)$.

(ii)⇒(iii). Let $\sum_{n=1}^\infty y_n^*$ be a weakly unconditionally Cauchy series in Y^*. This series allows us to create a linear map $\phi : \mathcal{R}_{bvc}(Y) \to \mathbb{R}$ through $\phi(y_n) = \sum_{n=1}^\infty \langle y_n^*, y_n\rangle$ for all $(y_n) \in \mathcal{R}_{bvc}(Y)$. The very definition of $\mathcal{R}_{bvc}(Y)$ ensures that for each $(y_n) \in \mathcal{R}_{bvc}(Y)$ there exists an absolutely convergent series $\sum_{k=1}^\infty z_k$ satisfying $\{y_n : n \in \mathbb{N}\} \subset \{\sum \alpha_k z_k : \|(\alpha_k)\|_\infty \le 1\}$. Then we have

$$|\phi(y_n)| \le \sum_{n=1}^\infty |\langle y_n^*, y_n\rangle| \le \sum_{k=1}^\infty \sum_{n=1}^\infty |\langle y_n^*, z_k\rangle|$$
$$\le \sup\{\sum_{n=1}^\infty |\langle y_n^*, y\rangle| : \|y\| \le 1\} \cdot \sum_{k=1}^\infty \|z_k\|.$$

Passing to the infimum we get

$$|\phi(y_n)| \le \sup\{\sum_{n=1}^\infty |\langle y_n^*, y\rangle| : \|y\| \le 1\} \cdot \|y_n\|_{bvc}.$$

This means that ϕ is bounded. The hypothesis (ii) alerts us to the fact that a map $\hat{T} : C_0(X) \to \mathcal{R}_{bvc}(Y)$ defined by $\hat{T}(x_n) = (Tx_n)$ for all $(x_n) \in C_0(X)$, is linear and bounded. Hence the composition $\phi \circ \hat{T} : C_0(X) \to \mathbb{R}$ is linear and bounded. Since $\phi \circ \hat{T}(x_n) = \sum_{n=1}^\infty \langle y_n^*, Tx_n\rangle = \sum_{n=1}^\infty \langle x_n, T^* y_n^*\rangle$ for all $(x_n) \in C_0(X)$ and the dual of $C_0(X)$ is the

space $\ell_1(X^*)$, we have $\sum_{n=1}^{\infty} \|T^* y_n^*\| < \infty$. The upshot of all this is that there must be a constant $C \geq 0$ such that regardless of the finite sequence (y_n^*) in Y^*, $\sum_{n=1}^{\infty} \|T^* y_n^*\| \leq C \, \|(y_n^*)\|_1^{\text{weak}}$. This gives that $\sum_{k=1}^{\infty} |\langle Tx_k, y_k^*\rangle| \leq C \cdot \|(x_k)\|_\infty \cdot \|(y_k^*)\|_1^{\text{weak}}$ regardless of the choice of finite sets $\{x_1, \cdots, x_n\} \subset X$ and $\{y_1^*, \cdots, y_n^*\} \subset Y^*$. That is $T \in \Pi_{1,\infty,1}(X, Y)$.
(iii)$\Rightarrow$(i). Let us take $(x_n) \in C_0(X)$. This sequence permits us to define an operator $U : \ell_1 \to X$ by $U(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n x_n$ for all $(\alpha_n) \in \ell_1$. The hypothesis (iii) guarantees that $T^* \in \Pi_1(Y^*, X^*)$ and so $U^*T^* \in \Pi_1(Y^*, \ell_\infty)$. The injectivity of ℓ_∞ makes that $U^*T^* \in \mathcal{I}_1(Y^*, \ell_\infty)$ and thus $TU \in \mathcal{I}_1(\ell_1, Y)$. Therefore TU admits a typical factorization $\kappa_Y TU : \ell_1 \xrightarrow{B} L^\infty(\mu) \xrightarrow{i_1} L^1(\mu) \xrightarrow{A} Y^{**}$, where μ is a finite regular Borel measure on some compact Hausdorff space Ω, i_1 is the formal identity, and A and B are bounded linear operators. As i_1 is weak*-weak continuous and absolutely summing, it has a representing measure m with bounded variation. A result due to Ryll-Nardzewski [4] provides an $L^1(\mu)$-valued measure $\tilde{m}$ with bounded variation for which $\operatorname{rg} \tilde{m} = i_1(B_{L^\infty(\mu)})$. Then $m_1 = A \circ \tilde{m}$ is a Y^{**}-valued measure with bounded variation so that $\{\kappa_Y T x_n : n \in \mathbb{N}\} = \{\kappa_Y TU e_n : n \in \mathbb{N}\} = \{A i_1 B e_n : n \in \mathbb{N}\} \subset C \operatorname{rg} m_1$ for some constant $C > 0$. This reveals that $\kappa_Y T \in \mathcal{R}_{bv}(X, Y^{**})$. We use the same argument as that of the bounded semivariation case to obtain $T \in \mathcal{R}_{bv}(X, Y)$. $\square$

Applying the above theorems we draw the following useful relationship between the space $\mathcal{R}$ and the space $\mathcal{R}_{bv}$.

Corollary. *Let $T \in \mathcal{L}(X, Y)$. Then the following statements are equivalent.*

(i) $T \in \mathcal{R}(X, Y)$.
(ii) *For any Banach space Z and any operator $S \in \mathcal{I}_\infty^*(Y, Z)$, we have $ST \in \mathcal{R}_{bv}(X, Z)$.*

Proof. (i)$\Rightarrow$(ii). We select any operator $S \in \mathcal{I}_\infty^*(Y, Z)$, where Z is any Banach space. Let $y_1, \cdots, y_n \in Y$ be given. Choose a finite sequence $(\alpha_k)_{k=1}^n$ such that $\|(\lambda_k)_{k=1}^n\|_{\ell_\infty^n} = 1$ and $\sum_{k=1}^n \lambda_k \|Sy_k\| = \sum_{k=1}^n \|Sy_k\|$. Next choose $z_1^*, \cdots, z_n^* \in B_{Z^*}$ so that $\langle z_k^*, Sy_k\rangle = \|Sy_k\|$ for $k = 1, \cdots, n$. Consider the composition of operators $\ell_\infty^n \xrightarrow{a} Y \xrightarrow{S}$

$Z \xrightarrow{b} \ell_\infty^n \xrightarrow{U} \ell_\infty^n$, where $a(\alpha_k) = \sum_{k=1}^n \alpha_k y_k$ for all $(\alpha_k)_{k=1}^n \in \ell_\infty^n$, $bz = (\langle z_k^*, z\rangle)_{k=1}^n$ for all $z \in Z$ and $U(\alpha_k) = (\lambda_k \alpha_k)_{k=1}^n$ for all $(\alpha_k)_{k=1}^n \in \ell_\infty^n$. Notice that $\|a\| = \|(y_k)\|_1^{\text{weak}}$, $\|b\| \le 1$ and $\imath_\infty(U) \le 1$. It follows that

$$\begin{aligned}\sum_{k=1}^n \|Sy_k\| = \sum_{k=1}^n \lambda_k \langle z_k^*, Sy_k\rangle = \sum_{k=1}^n \lambda_k \langle b^* e_k, Sae_k\rangle = \sum_{k=1}^n \langle e_k, UbSae_k\rangle \\ = \operatorname{tr}(UbSa) \le \imath_\infty^*(S)\,\|a\|\,\|b\|\,\imath_\infty(U) \le \imath_\infty^*(S)\,\|(y_k)\|_1^{\text{weak}}.\end{aligned}$$

As a result $S \in \Pi_1(Y,Z)$ with $\pi_1(S) \le \imath_\infty^*(S)$. Now let us take $(x_n) \in C_0(X)$. The hypothesis (i) enables us to invoke theorem 1 to get that $T \in \mathcal{R}_c(X,Y)$ and hence $(Tx_n) \in \mathcal{R}_c(Y)$. According to proposition 1.4 of [11], there exists an unconditionally convergent series $\sum_{k=1}^\infty y_k$ in Y such that $\{Tx_n : n \in \mathbb{N}\} \subset \{\sum_{k=1}^\infty \alpha_k y_k : \|(\alpha_k)\|_\infty \le 1\}$. Therefore $\{STx_n : n \in \mathbb{N}\} \subset \{\sum_{k=1}^\infty \alpha_k Sy_k : \|(\alpha_k)\|_\infty \le 1\}$, and furthermore $\sum_{k=1}^\infty \|Sy_k\| < \infty$ because $S \in \Pi_1(Y,Z)$. This gives that $ST \in \mathcal{R}_{bvc}(X,Z)$. We make use of theorem 2 to obtain that $ST \in \mathcal{R}_{bv}(X,Z)$.
(ii)⇒(i). Let us take any $S \in \Pi_1(Y,\ell_1)$. Suppose E and F are finite dimensional Banach spaces, and let $a \in \mathcal{L}(E,Y)$, $b \in \mathcal{L}(\ell_1,F)$,$U \in \mathcal{L}(F,E)$. From a result of Grothendieck [6] we get $|\operatorname{tr}(UbSa)| \le \imath_1(UbSa)$ which thanks to a theorem of Persson and Pietsch [10] lead to $|\operatorname{tr}(UbSa)| \le \pi_1(Sa)\,\imath_\infty(Ub) \le \pi_1(S)\,\|a\|\,\|b\|\,\imath_\infty(U)$. It follows that $S \in \mathcal{I}_\infty^*(Y,\ell_1)$ and $\imath_\infty^*(S) \le \pi_1(S)$. Then the hypothesis (ii) tells us that $ST \in \mathcal{R}_{bv}(X,\ell_1)$. Appealing to theorem 2 we have that $ST \in \Pi_{1,\infty,1}(X,\ell_1)$ and so $(ST)^* \in \Pi_1(\ell_\infty, X^*)$. Writing $(ST)^* e_n = x_n^*$, we find $\sum_{n=1}^\infty \|x_n^*\| < \infty$, from which we drive that $ST \in \mathcal{N}(X,\ell_1)$, where $ST(x) = (\langle x_n^*, x\rangle)$ for all $x \in X$. Thus we can define the operator $\mathcal{T} : \Pi_1(Y,\ell_1) \to \mathcal{N}(X,\ell_1)$ assigning to every absolutely summing operator S the nuclear operator ST. Suppose (S_n, S_nT) is a sequence that converges to an element (S,R) in $\Pi_1(Y,\ell_1) \times \mathcal{N}(X,\ell_1)$, that is $\lim_{n\to\infty} \pi_1(S_n - S) = 0$ and $\lim_{n\to\infty} \nu(S_nT - R) = 0$. Then

$$\begin{aligned}\pi_1(R - ST) &\le \pi_1(R - S_nT) + \pi_1(S_nT - ST) \\ &\le \nu(R - S_nT) + \|T\|\,\pi_1(S_n - S).\end{aligned}$$

This yields that $R = ST$ and so the graph of $\mathcal{T}$ is closed. The closed graph theorem provides us with a constant $C > 0$ such that $\nu(ST) \le C\,\pi_1(S)$ for each $S \in \Pi_1(Y,\ell_1)$. $(*)$

Now let $(u_i^*)_{i=1}^m$ and $(v_j^*)_{j=1}^n$ be finite sequences in Y^* satisfying $\sum_{i=1}^m |\langle u_i^*, y\rangle| \leq \sum_{j=1}^n |\langle v_j^*, y\rangle|$ for all $y \in Y$. $(**)$
We use these sequences to define operators $A : Y \to \ell_1^m$ and $B : Y \to \ell_1^n$ by $Ay = (\langle u_i^*, y\rangle)_{i=1}^m$ for all $y \in Y$ and $By = (\langle v_j^*, y\rangle)_{j=1}^n$ for all $y \in Y$, respectively. Take any finite set $\{y_1, \cdots, y_n\} \subset Y$. Then it follows from condition $(**)$ that

$$\sum_{k=1}^n \|Ay_k\| \leq \sum_{k=1}^n \|By_k\| \leq \pi_1(B) \cdot \sup\{\sum_{k=1}^n |\langle y^*, y_k\rangle| : \|y^*\| \leq 1\}.$$

and hence $\pi_1(A) \leq \pi_1(B)$. Note that $ATx = (\langle T^* u_i^*, x\rangle)_{i=1}^m$ for all $x \in X$ and $AT \in \mathcal{N}(X, \ell_1)$. It takes an appeal to condition $(*)$ to see that

$$\sum_{i=1}^m \|T^* u_i^*\| = \nu(AT) \leq C\,\pi_1(A) \leq C\,\pi_1(B) \leq C\,\nu(B) = C\,\sum_{i=1}^n \|v_i^*\|.$$

We summon up theorem 1 to conclude that $T \in \mathcal{R}(X, Y)$. □

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