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THE DENJOY-STIELTJES EXTENSION OF THE BOCHNER, DUNFORD AND PETTIS INTEGRALS

Chun-Kee Park

Abstract. In this paper we introduce the concepts of Denjoy-Stieltjes-Dunford, Denjoy-Stieltjes-Pettis and Denjoy-Stieltjes-Bochner integrals of Banach-valued functions and then prove some properties of them.

1. Introduction

The Denjoy integral of a real-valued function which is an extension of the Lebesgue integral was studied by some authors ([1],[2],[3],[7]). In [5] we introduced the Denjoy-Stieltjes integral which is the generalization of the Denjoy integral and obtained some properties of the Denjoy-Stieltjes integral. R.A.Gordon[2], J.L.Gamez and J.Mendoza[1] studied the Denjoy extension of the Bochner, Pettis and Dunford integrals which is defined by the Denjoy integral. In this paper we deal with the Denjoy-Stieltjes extension of the Bochner, Pettis and Dunford integrals which is the generalization of the Denjoy extension of the Bochner, Pettis and Dunford integrals. We first define Denjoy-Stieltjes-Dunford, Denjoy-Stieltjes-Pettis and Denjoy-Stieltjes-Bochner integrals of Banach-valued functions using the Denjoy-Stieltjes integral and then prove some properties of them.

2. Preliminaries

We give some definitions and results to be used in this paper. Throughout this paper, X denotes a real Banach space and X^* its

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dual.

DEFINITION 2.1[3]. Let $F : [a, b] \to X$ and let $E \subset [a, b]$.

(a) The function F is BV on E if $V(F, E) = \sup \left\{ \sum_{i=1}^{n} \|F(d_i) - F(c_i)\| \right\}$ is finite where the supremum is taken over all finite collections $\{[c_i, d_i] : 1 \le i \le n\}$ of nonoverlapping intervals that have endpoints in E.

(b) The function F is AC on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \|F(d_i) - F(c_i)\| < \epsilon$ whenever $\{[c_i, d_i] : 1 \le i \le n\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$.

(c) The function F is BVG on E if E can be expressed as a countable union of sets on each of which F is BV.

(d) The function F is ACG on E if F is continuous on E and if E can be expressed as a countable union of sets on each of which F is AC.

DEFINITION 2.2[2]. Let $F : [a, b] \to X$ and let $t \in (a, b)$. A vector z in X is the approximate derivative of F at t if there exists a measurable set $E \subset [a, b]$ that has t as a point of density such that $\lim_{\substack{s \to t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = z$. We will write $F'_{ap}(t) = z$.

A function $f : [a, b] \to \mathbb{R}$ is Denjoy integrable on $[a, b]$ if there exists an ACG function $F : [a, b] \to \mathbb{R}$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$. The function f is Denjoy integrable on the set $E \subset [a, b]$ if $f\chi_E$ is Denjoy integrable on $[a, b]$.

DEFINITION 2.3[2]. (a) A function $f : [a, b] \to X$ is Denjoy-Dunford integrable on $[a, b]$ if for each $x^* \in X^*$ the function $x^* f$ is Denjoy integrable on $[a, b]$ and if for every interval I in $[a, b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^* f$ for all $x^* \in X^*$.

(b) A function $f : [a, b] \to X$ is Denjoy-Pettis integrable on $[a, b]$ if f is Denjoy-Dunford integrable on $[a, b]$ and if $x_I^{**} \in X$ for every interval I in $[a, b]$.

(c)A function $f : [a,b] \to X$ is Denjoy-Bochner integrable on $[a,b]$ if there exists an ACG function $F : [a,b] \to X$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ab} = f$ almost everywhere on $[a,b]$.

DEFINITION 2.4[5]. Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function and let $E \subset [a,b]$.

(a) The function F is BV with respect to α on E if $V(F,\alpha,E) = \sup\left\{\sum_{i=1}^{n} \|F(d_i) - F(c_i)\| \frac{\alpha(d_i)-\alpha(c_i)}{d_i - c_i}\right\}$ is finite where the supremum is taken over all finite collections $\{[c_i,d_i] : 1 \leq i \leq n\}$ of nonoverlapping intervals that have endpoints in E.

(b) The function F is AC with respect to α on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \|F(d_i) - F(c_i)\| < \epsilon$ whenever $\{[c_i,d_i] : 1 \leq i \leq n\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}[\alpha(d_i) - \alpha(c_i)] < \delta$.

(c) The function F is BVG with respect to α on E if E can be expressed as a countable union of sets on each of which F is BV with respect to α.

(d) The function F is ACG with respect to α on E if F is continuous on E and if E can be expressed as a countable union of sets on each of which F is AC with respect to α.

THEOREM 2.5[5]. *Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. Then F is BV on E if and only if F is BV with respect to α on E.*

THEOREM 2.6[5]. *Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. Then F is AC on E if and only if F is AC with respect to α on E.*

3. Denjoy-Stieltjes integral

In [5] we introduced the Denjoy-Stieltjes integral and obtained some results for the integral. In this section we give another result.

DEFINITION 3.1[5]. Let $F : [a,b] \to X$ and let $t \in (a,b)$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. A vector $z \in X$ is the approximate derivative of F with respect to α at t if there exists a measurable set $E \subset [a,b]$ that has t as a point of density such that $\lim_{\substack{s \to t \\ s \in E}} \dfrac{F(s) - F(t)}{\alpha(s) - \alpha(t)} = z$. We will write $F'_{\alpha,ap}(t) = z$.

We note that $F'_{ap}(t) = F'_{\alpha,ap}(t) \cdot \alpha'(t)$ for each $t \in (a,b)$.

DEFINITION 3.2[5]. Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. A function $f : [a,b] \to \mathbb{R}$ is Denjoy-Stieltjes integrable with respect to α on $[a,b]$ if there exists an ACG function $F : [a,b] \to \mathbb{R}$ with respect to α such that $F'_{\alpha,ap} = f$ almost everywhere on $[a,b]$. The function f is Denjoy-Stieltjes integrable with respect to α on a set $E \subset [a,b]$ if $f\chi_E$ is Denjoy-Stieltjes integrable with respect to α on $[a,b]$.

THEOREM 3.3[5]. *Let $f : [a,b] \to \mathbb{R}$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. Then f is Denjoy-Stieltjes integrable with respect to α on E if and only if $\alpha' f$ is Denjoy integrable on E.*

THEOREM 3.4. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. If $f : [a,b] \to \mathbb{R}$ is Denjoy-Stieltjes integrable with respect to α on each interval $[c,d] \subseteq (a,b)$ and $\int_c^d f \, d\alpha$ converges to a finite limit as $c \to a^+$ and $d \to b^-$, then f is Denjoy-Stieltjes integrable with respect to α on $[a,b]$ and $\int_a^b f \, d\alpha = \lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d f \, d\alpha$.*

Proof. Since $f : [a,b] \to \mathbb{R}$ is Denjoy-Stieltjes integrable with respect to α on each interval $[c,d] \subseteq (a,b)$, by Theorem 3.3 $\alpha' f : [a,b] \to \mathbb{R}$ is Denjoy integrable on each interval $[c,d] \subseteq (a,b)$ and $\int_c^d f \, d\alpha = \int_c^d \alpha' f$ for each interval $[c,d] \subseteq (a,b)$. Hence $\lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d \alpha' f = \lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d f \, d\alpha$ exists by hypothesis. By [3, Theorem 15.12], $\alpha' f$ is Denjoy integrable on

$[a,b]$ and $\displaystyle\int_a^b \alpha' f = \lim_{\substack{c\to a^+\\ d\to b^-}} \int_c^d \alpha' f$. By Theorem 3.3, f is Denjoy-Stieltjes integrable with respect to α on $[a,b]$ and $\displaystyle\int_a^b f\, d\alpha = \lim_{\substack{c\to a^+\\ d\to b^-}} \int_c^d f\, d\alpha$. □

4. Denjoy-Stieltjes extension of the Bochner, Pettis and Dunford integrals

We introduce Denjoy-Stieltjes-Bochner, Denjoy-Stieltjes-Pettis and Denjoy-Stieltjes-Dunford integrals and investigate some properties of those integrals.

Definition 4.1. Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$.

(a) $f : [a,b] \to X$ is Denjoy-Stieltjes-Dunford integrable with respect to α on $[a,b]$ if for each $x^* \in X^*$ x^*f is Denjoy-Stieltjes integrable with respect to α on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector $x_I^{**} \in X^{**}$ such that $x_I^{**}(x^*) = \int_I x^*f\, d\alpha$ for all $x^* \in X^*$.

(b) $f : [a,b] \to X$ is Denjoy-Stieltjes-Pettis integrable with respect to α on $[a,b]$ if f is Denjoy-Stieltjes -Dunford integrable with respect to α on $[a,b]$ and if $x_I^{**} \in X$ for every interval I in $[a,b]$.

(c) $f : [a,b] \to X$ is Denjoy-Stieltjes-Bochner integrable with respect to α on $[a,b]$ if there exists an ACG function $F : [a,b] \to X$ with respect to α such that F is approximately differentiable with respect to α almost everywhere on $[a,b]$ and $F'_{\alpha,ap} = f$ almost everywhere on $[a,b]$.

$f : [a,b] \to X$ is integrable in one of the above senses on the set $E \subseteq [a,b]$ if $f\chi_E$ is integrable in that sense on $[a,b]$.

Theorem 4.2. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. Then $f : [a,b] \to X$ is Denjoy-Stieltjes-Bochner integrable with respect to α on E if and only if $\alpha' f : [a,b] \to X$ is Denjoy-Bochner integrable on E.*

Proof. If $f : [a,b] \to X$ is Denjoy-Stieltjes-Bochner integrable with respect to α on E, then there exists an ACG function $F : [a,b] \to X$ with respect to α such that F is approximately differentiable with respect to α almost everywhere on $[a,b]$ and $F'_{\alpha,ap} = f\chi_E$ almost everywhere on $[a,b]$. By Theorem 2.6, F is ACG. F is also approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = F'_{\alpha,ap}\alpha' = \alpha' f\chi_E$ almost everywhere on $[a,b]$. Hence $\alpha' f$ is Denjoy-Bochner integrable on E.

Conversely, if $\alpha' f : [a,b] \to X$ is Denjoy-Bochner integrable on E, then there exists an ACG function $F : [a,b] \to X$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = \alpha' f\chi_E$ almost everywhere on $[a,b]$. By Theorem 2.6, F is ACG with respect to α on $[a,b]$. F is also approximately differentiable with respect to α almost everywhere on $[a,b]$ and $F'_{\alpha,ap} = \frac{1}{\alpha'}F_{ap} = \frac{1}{\alpha'}\alpha' f\chi_E = f\chi_E$ almost everywhere on $[a,b]$. Hence f is Denjoy-Stieltjes-Bochner integrable with respect to α on E. □

The following corollary is obtained from Theorem 4.2 and [2, Theorem 28].

Corollary 4.3. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. If $f : [a,b] \to X$ is Denjoy-Stieltjes-Bochner integrable with respect to α on $[a,b]$, then each perfect set in $[a,b]$ contains a portion on which $\alpha' f$ is Bochner integrable.*

Theorem 4.4. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subseteq [a,b]$. Then $f : [a,b] \to X$ is Denjoy-Stieltjes-Dunford integrable with respect to α on E if and only if $\alpha' f : [a,b] \to X$ is Denjoy-Dunford integrable on E.*

Proof. If $f : [a,b] \to X$ is Denjoy-Stieltjes-Dunford integrable with respect to α on E, then for each $x^* \in X^*$ x^*f is Denjoy-Stieltjes integrable with respect to α on E and for every interval I in $[a,b]$ there exists a vector $x_I^{**} \in X^{**}$ such that $x_I^{**}(x^*) = \int_I x^* f\chi_E \, d\alpha$ for all $x^* \in X^*$. By Theorem 3.3, for each $x^* \in X^*$ $\alpha'(x^*f) = x^*(\alpha' f)$ is Denjoy integrable on E and $x_I^{**}(x^*) = \int_I x^* f\chi_E \, d\alpha = \int_I x^*(\alpha' f\chi_E)$ for all $x^* \in X^*$. Hence $\alpha' f : [a,b] \to X$ is Denjoy-Dunford integrable on E.

Conversely, if $\alpha' f : [a,b] \to X$ is Denjoy-Dunford integrable on E, then for each $x^* \in X^*$ $x^*(\alpha' f) = \alpha'(x^*f)$ is Denjoy integrable on E and for every interval I in $[a,b]$ there exists a vector $x_I^{**} \in X^{**}$ such that $x_I^{**}(x^*) = \int_I x^*(\alpha' f \chi_E)$ for all $x^* \in X^*$. By Theorem 3.3, for each $x^* \in X^*$ $x^* f$ is Denjoy-Stieltjes integrable with respect to α on E and $x_I^{**}(x^*) = \int_I x^*(\alpha' f \chi_E) = \int_I \alpha'(x^* f \chi_E) = \int_I x^* f \chi_E \; d\alpha$ for all $x^* \in X^*$. Hence $f : [a,b] \to X$ is Denjoy-Stieltjes-Dunford integrable with respect to α on E. □

Corollary 4.5. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subseteq [a,b]$. Then $f : [a,b] \to X$ is Denjoy-Stieltjes-Pettis integrable with respect to α on E if and only if $\alpha' f : [a,b] \to X$ is Denjoy-Pettis integrable on E.*

Proof. The proof is similar to Theorem 4.4. □

The following Corollary is obtained from Theorem 3.3, Theorem 4.4 and [1, Theorem 3].

Corollary 4.6. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subseteq [a,b]$. Then $f : [a,b] \to X$ is Denjoy-Stieltjes-Dunford integrable with respect to α on E if and only if $x^* f$ is Denjoy-Stieltjes integrable with respect to α on E for all $x^* \in X^*$.*

Theorem 4.7. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. If $f : [a,b] \to \mathbb{R}$ is Denjoy-Stieltjes-Dunford integrable with respect to α on $[a,t]$ for all $t \in [a,b]$ and for each $x^* \in X^*$ $\lim_{t \to b} \int_a^t x^* f \; d\alpha$ exists, then f is Denjoy-Stieltjes-Dunford integrable with respect to α on $[a,b]$ and $< x^*, (DSD) \int_a^b f \; d\alpha >= \lim_{t \to b} < x^*, (DSD) \int_a^t f \; d\alpha >$ for each $x^* \in X^*$.*

Proof. If $f : [a,b] \to X$ is Denjoy-Stieltjes-Dunford integrable with respect to α on $[a,t]$ for all $t \in [a,b)$ and for each $x^* \in X^*$ $\lim_{t \to b} \int_a^t x^* f \; d\alpha$ exists, then by Theorem 3.4 $x^* f$ is Denjoy-Stieltjes integrable with respect to α on $[a,b]$ and $\int_a^b x^* f \; d\alpha = \lim_{t \to b} \int_a^t x^* f \; d\alpha$ for all $x^* \in X^*$. Take $c \in [a,b)$ and any sequence (t_n) in $[a,b)$ convergent to *b*. Define $L_c(x^*) = \lim_{n\to\infty} \int_c^{t_n} x^* f \; d\alpha = \lim_{n\to\infty} < x^*, (DSD) \int_c^{t_n} f \; d\alpha >$ for each $x^* \in X^*$. By the uniform bounded principle, the linear functional L_c is continuous on X^*, that is, $L_c \in X^{**}$. Hence it is immediate that f is Denjoy-Stieltjes-Dunford integrable with respect to α on $[a, b]$. Taking $c = a$, we get

$$
\begin{aligned}
< x^*, (DSD) \int_a^b f \; d\alpha > &= \int_a^b x^* f \; d\alpha \\
&= \lim_{t\to b} \int_a^t x^* f \; d\alpha \\
&= \lim_{t\to b} < x^*, (DSD) \int_a^t f \; d\alpha >
\end{aligned}
$$

for each $x^* \in X^*$. □

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Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea