NOTE ON THE FUZZY PROXIMITY SPACES

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Abstract. This paper is devoted to the study of the role of fuzzy proximity spaces. We define a fuzzy K-proximity space, a fuzzy R-proximity space and prove some of its properties. Furthermore, we discuss the topological structure based on these fuzzy K-proximity and fuzzy R-proximity.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [11] in 1965. This idea was used by Chang [2], who in 1968 defined fuzzy topological spaces, and by Lowen [6], who in 1974 defined fuzzy uniform spaces. More recently, Katsaras [3], who in 1979, defined fuzzy proximities, on the base of the axioms suggested by Efremović [8].

In this paper we propose some generalization of the concept of the fuzzy proximity, which we call a "fuzzy K-proximity" and a "fuzzy R-proximity". We also try to examine some of its properties and characterize the topological structure based on these fuzzy K-proximity and fuzzy R-proximity.

2. Preliminaries

As a preparation, we briefly review some basic definitions concerning a fuzzy proximity space. Throughout this paper, $X$ is reserved to denote a nonempty set and let $I^X$ be the collection of all mappings from $X$ to the unit closed interval $I = [0, 1]$ of the real line. A member

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\( \mu \) of \( I^X \) is called a fuzzy set of \( X \). For any \( \mu, \rho \in I^X \), the join \( \mu \lor \rho \), and the meet \( \mu \land \rho \) of \( \mu \) and \( \rho \) defined as followings: For any \( x \in X \),

\[
(\mu \lor \rho)(x) = \sup \{\mu(x), \rho(x)\} \quad \text{and} \quad (\mu \land \rho)(x) = \inf \{\mu(x), \rho(x)\},
\]

respectively. And \( \mu \leq \rho \) if for each \( x \in X, \mu(x) \leq \rho(x) \). The complement \( \mu' \) of a fuzzy set \( \mu \) in \( X \) is \( 1 - \mu \) defined by \( \mu'(x) = (1 - \mu)(x) = 1 - \mu(x) \) for each \( x \in X \). 0 and 1 denote constant functions mapping all of \( X \) to 0 and 1, respectively. Now we give the definitions of a fuzzy topology and a closure operator.

**Definition 2.1.** A fuzzy topology on \( X \) is a subset \( \alpha \) of \( I^X \) which satisfies the following conditions:

1. (FT1) \( 0, 1 \in \alpha \).
2. (FT2) If \( \mu, \rho \in \alpha \), then \( \mu \land \rho \in \alpha \).
3. (FT3) If \( \mu_i \in \alpha \) for each \( i \in A \), then \( \sup_{i \in A} \mu_i \in \alpha \).

The pair \( (X, \alpha) \) is called a fuzzy topological space, or fts for short.

**Definition 2.2.** A map \( \mu \mapsto \text{cl}(\mu) \), from \( I^X \) into \( I^X \), is said to be a closure operator if it satisfies the following conditions:

1. (C1) \( \mu \leq \text{cl}(\mu) \).
2. (C2) \( \text{cl}(\text{cl}(\mu)) = \text{cl}(\mu) \).
3. (C3) \( \text{cl}(\mu \lor \rho) = \text{cl}(\mu) \lor \text{cl}(\rho) \).
4. (C4) \( \text{cl}(0) = 0 \).

Given a closure operator on \( I^X \), the collection

\[
\{\mu \in I^X \mid \text{cl}(1 - \mu) = 1 - \mu\}
\]

is a fuzzy topology on \( X \).

In the following we first define a fuzzy proximity space and a fuzzy point. Let \( \delta \) be a binary relation on \( I^X \), i.e., \( \delta \subset I^X \times I^X \). The facts that \( (\mu, \rho) \in \delta \) and \( (\mu, \rho) \notin \delta \) are denoted by \( \mu \delta \rho \) and \( \mu \tilde{\delta} \rho \), respectively.

**Definition 2.3.** A binary relation \( \delta \) on \( I^X \) is called a fuzzy proximity if \( \delta \) satisfies the following conditions:

1. (FP1) \( \mu \delta \rho \) implies \( \rho \delta \mu \).
2. (FP2) \( (\mu \lor \rho) \delta \sigma \) if and only if \( \mu \delta \sigma \) or \( \rho \delta \sigma \).
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(FP3) $\mu \delta \rho$ implies $\mu \neq 0$ and $\rho \neq 0$.

(FP4) $\mu \delta \rho$ implies that there exists a $\sigma \in I^X$ such that $\mu \delta \sigma$ and $(1 - \sigma) \delta \rho$.

(FP5) $\mu \land \rho \neq 0$ implies $\mu \delta \rho$.

The pair $(X, \delta)$ is called a fuzzy proximity space.

**Definition 2.4.** A fuzzy set in $X$ is called a fuzzy point if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at $x$ is $\gamma (0 < \gamma < 1)$, we denote this fuzzy point by $x_\gamma$, where the point $x$ is called its support.

**Definition 2.5.** The fuzzy point $x_\gamma$ is said to be contained in a fuzzy set $\mu$, or to belong to $\mu$, denoted by $x_\gamma \in \mu$, if $\gamma < \mu(x)$. Evidently, every fuzzy set $\mu$ can be expressed as the union of all the fuzzy points which belong to $\mu$.

**3. Fuzzy K-Proximity**

We define a fuzzy K-proximity space and we investigate some properties of this structure.

**Definition 3.1.** A binary relation $\delta$ on $I^X$ is called a fuzzy K-proximity if $\delta$ satisfies the following conditions:

(FK1) $x_\gamma \delta (\mu \lor \rho)$ if and only if $x_\gamma \delta \mu$ or $x_\gamma \delta \rho$.

(FK2) $x_\gamma \delta 0$ for all $x_\gamma$.

(FK3) $x_\gamma \in \mu$ implies $x_\gamma \delta \mu$.

(FK4) $x_\gamma \delta \mu$ implies that there exists a $\rho \in I^X$ such that $x_\gamma \delta \rho$ and $y_\gamma \delta \mu$ for all $y_\gamma \in (1 - \rho)$.

The pair $(X, \delta)$ is called a fuzzy K-proximity space.

One can easily show that the fuzzy proximity on $I^X$ implies the fuzzy K-proximity on $I^X$.

**Theorem 3.2.** Every fuzzy proximity on $I^X$ implies the fuzzy K-proximity on $I^X$.

**Proof.** (FP1) and (FP2) imply (FK1), (FP3) implies (FK2), and (FP5) implies (FK3). If $\mu = \{x_\gamma\}$ and $\mu \delta \rho$, then (FP4) there exists a
\( \sigma \in I^X \) with \( x_\gamma \delta \sigma \), and \((1 - \sigma) \delta \rho \). Hence for each \( y_\gamma \in (1 - \sigma) \), we have \( y_\gamma \delta \rho \). This means that (FP1) and (FP4) implies (FK4). \( \square \)

Now we shall introduce the fuzzy proximity \( \delta_1 \) from the fuzzy K-proximity \( \delta \) replacing the axiom (FK4) in the fuzzy K-proximity by the stronger one.

**Definition 3.3.** A binary relation \( \delta \) on \( I^X \) is called the fuzzy proximity if \( \delta \) satisfies the axioms (FP1), (FP2), (FP3) in the Definition 2.3, and (FP4\textsuperscript{\prime}) For each \( \sigma \in I^X \) there is a fuzzy point \( x_\gamma \) such that either \( x_\gamma \delta \mu, x_\gamma \delta \sigma \) or \( x_\gamma \delta \rho, x_\gamma \delta (1 - \sigma) \), then we have \( x_\gamma \delta \mu \) and \( x_\gamma \delta \rho \).

**Definition 3.4.** In a fuzzy K-proximity space \((X, \delta)\), let \( \delta_1 \) be a binary relation on \( I^X \) defined as follows: For each \( \mu, \rho \in I^X \), \( \mu \delta_1 \rho \) if and only if there is a fuzzy point \( x_\gamma \) such that \( x_\gamma \delta \mu \) and \( x_\gamma \delta \rho \).

**Theorem 3.5.** The binary relation \( \delta_1 \) on \( I^X \) defined in Definition 3.4 is the fuzzy proximity.

**Proof.** We will show that \( \delta_1 \) satisfies (FP1) \( \sim \) (FP5).

(FP1) It is clear that \( \mu \delta_1 \rho \) implies \( \rho \delta_1 \mu \).

(FP2)

\[
(\mu \lor \rho) \delta_1 \sigma \iff \exists \text{ a fuzzy point } x_\gamma \text{ such that } x_\gamma \delta (\mu \lor \rho) \text{ and } x_\gamma \delta \sigma
\]

\[
\iff (x_\gamma \delta \mu \text{ or } x_\gamma \delta \rho) \text{ and } x_\gamma \delta \rho
\]

\[
\iff (x_\gamma \delta \mu, x_\gamma \delta \sigma) \text{ or } (x_\gamma \delta \rho, x_\gamma \delta \sigma)
\]

\[
\iff \mu \delta_1 \sigma \text{ or } \rho \delta_1 \sigma.
\]

(FP3)

\[
\mu \delta_1 \rho \implies \exists \text{ a fuzzy point } x_\gamma \text{ such that } x_\gamma \delta \mu \text{ and } x_\gamma \delta \rho
\]

\[
\implies \mu \neq 0 \text{ and } \rho \neq 0.
\]

(FP4) Suppose that for each \( \sigma \in I^X \), \( \mu \delta_1 \sigma \) or \( \rho \delta_1 (1 - \sigma) \). Hence for some fuzzy point \( x_\gamma \) we have either \( x_\gamma \delta \mu, x_\gamma \delta \sigma \) or \( x_\gamma \delta \rho, x_\gamma \delta (1 - \sigma) \), therefore by (FP4\textsuperscript{\prime}) \( x_\gamma \delta \mu \) and \( x_\gamma \delta \rho \), that is, \( \mu \delta_1 \rho \).
\(\mu \wedge \rho \neq 0 \implies \exists \) a fuzzy point \(x_\gamma\) such that \(x_\gamma \in \mu\) and \(x_\gamma \in \rho\)
\[\implies \ x_\gamma \delta \mu \text{ and } x_\gamma \delta \rho\]
\[\implies \mu \delta_1 \rho.\]

\[\Box\]

In what follows we introduce some properties of the fuzzy K-proximity.

**Lemma 3.6.** If \(x_\gamma \delta \mu\) and \(\mu \leq \rho\), then \(x_\gamma \delta \rho\).

**Proof.** By (FK1) \(x_\gamma \delta \mu \implies x_\gamma \delta (\mu \lor \rho) \implies x_\gamma \delta \rho.\) \(\Box\)

**Theorem 3.7.** In the fuzzy K-proximity space \((X, \delta)\) if \(\mu^\delta\) is defined to be a set \(\bigvee \{x_\gamma \mid x_\gamma \delta \mu \text{ and } x_\gamma \text{ is a fuzzy point in } X\}\) for each fuzzy set \(\mu\) in \(X\), then \(\delta\) is a closure operator. Hence we can introduce the fuzzy topology \(T(\delta)\) on \(X\) by \(\delta\).

**Proof.** Since the other axioms are easily verified, it suffices to show that \(\delta\) satisfies (C2). So, we assume that \(x_\gamma \overline{\delta} \mu\). Then by (FK4) there exists a \(\rho \in I^X\) such that \(x_\gamma \overline{\delta} \rho\) and \(y_\gamma \overline{\delta} \mu\) for all \(y_\gamma \in (1 - \rho)\). If \(z_\gamma \in \mu^\delta\), then \(z_\gamma \delta \mu\). Hence \(z_\gamma \in \rho\), that is \(\mu^\delta \leq \rho\). Since \(x_\gamma \overline{\delta} \rho\) we have \(x_\gamma \overline{\delta} \mu^\delta\). This means that \(x_\gamma \in \mu^\delta\) implies \(x_\gamma \in \mu^\delta\) or \(\mu^\delta \subset \mu^\delta\). Therefore \(\mu^\delta = \mu^\delta\). \(\Box\)

**Theorem 3.8.** Let \((X, \alpha)\) be a fuzzy topological space. If a binary relation \(\delta\) is defined by \(x_\gamma \delta \mu\) if and only if \(x_\gamma \in \text{cl}(\mu)\), then \(\delta\) is a fuzzy K-proximity on \(I^X\) and the fuzzy topology \(T(\delta)\) induced by \(\delta\) is the given topology \(\alpha\).

**Proof.** Now we will show that \(\delta\) satisfies (FK1) \(\sim\) (FK4).

\[\text{FK1}\]
\[x_\gamma \delta (\mu \lor \rho) \iff x_\gamma \in \text{cl}(\mu \lor \rho)\]
\[\iff x_\gamma \in \text{cl}(\mu) \lor x_\gamma \in \text{cl}(\rho)\]
\[\iff x_\gamma \delta \mu \text{ or } x_\gamma \delta \rho.\]
(FK2) \[ cl(0) = 0 \implies x\gamma \bar{0} \text{ for all } x\gamma. \]

(FK3) \[ x\gamma \in \mu \implies x\gamma \in cl(\mu) \implies x\gamma \delta \mu. \]

(FK4) \[ x\gamma \delta \mu \iff x\gamma \notin cl(\mu) \iff x\gamma \notin cl(cl(\mu)) \iff \text{if } cl(\mu) = \rho, \text{ then } x\gamma \delta \rho \text{ and } y\gamma \delta \mu \text{ for all } y\gamma \in (1 - cl(\mu)). \]

Since \( x\gamma \in cl(\mu) \iff x\gamma \delta \mu \iff x\gamma \in \mu \delta \), we have \( cl(\mu) = \mu \delta \), that is, \( T(\delta) = \alpha \). \( \square \)

**Theorem 3.9.** The fuzzy topological space \( X \) is \( T_1 \) if and only if there is a fuzzy K-proximity \( \delta \) on \( I^X \) satisfying the following condition:

(FK5) \( x\gamma \delta \{ y\gamma \} \implies x\gamma = y\gamma. \)

**Proof.** Assume \( X \) is \( T_1 \). Then there is a binary relation \( \delta \) on \( I^X \) satisfying conditions (FK1) \( \sim \) (FK4). So, \( x\gamma \in \mu \delta \iff x\gamma \delta \mu \). Hence \( x\gamma \delta \{ y\gamma \} \implies x\gamma \in \{ y\gamma \} \delta = \{ y\gamma \}, \) since \( X \) is \( T_1 \). That is, \( x\gamma = y\gamma \).

Conversely, if \( x\gamma \delta \{ y\gamma \} \) implies that \( x\gamma = y\gamma \) then \( \{ y\gamma \} \delta = \{ y\gamma \}, \) that is, \( X \) is \( T_1 \). \( \square \)

**Lemma 3.10.** \( x\gamma \delta \{ y\gamma \} \) and \( y\gamma \delta \mu \implies x\gamma \delta \mu. \)

**Proof.**

\( x\gamma \delta \mu \implies \exists \rho \text{ such that } x\gamma \delta \rho \text{ and } z\gamma \delta \mu \text{ for all } z\gamma \in (1 - \rho) \)

\[ \implies y\gamma \notin \rho( \text{ if } y\gamma \in \rho \text{ then } x\gamma \delta \{ y\gamma \}, \ y\gamma \in \rho \text{ so we have } x\gamma \delta \rho) \]

\[ \implies y\gamma \in (1 - \rho), \text{ that is, } y\gamma \delta \mu. \]

It is a contradiction. \( \square \)
4. Fuzzy R-Proximity

We introduce a fuzzy R-proximity and we prove that some of properties of this notion.

**Definition 4.1.** A binary relation $\delta$ on $I^X$ is called a fuzzy R-proximity if $\delta$ satisfies the following conditions:

1. **(FR1)** $\mu \delta \rho$ implies $\rho \delta \mu$.
2. **(FR2)** $(\mu \lor \rho) \delta \sigma$ if and only if $\mu \delta \sigma$ or $\rho \delta \sigma$.
3. **(FR3)** $\mu \delta \rho$ implies $\mu \neq 0$ and $\rho \neq 0$.
4. **(FR4)** $x_\gamma \delta \mu$ implies that there exists a $\rho \in I^X$ such that $x_\gamma \delta \rho$ and $(1 - \rho) \delta \mu$.
5. **(FR5)** $\mu \land \rho \neq 0$ implies $\mu \delta \rho$.

The pair $(X, \delta)$ is called a fuzzy R-proximity space.

**Theorem 4.2.** In a fuzzy R-proximity space $(X, \delta)$ if $\mu^\delta$ is defined to be a set $\bigvee \{x_\gamma \mid x_\gamma \delta \mu \text{ and } x_\gamma \text{ is a fuzzy point in } X\}$ for each fuzzy set $\mu$ in $X$, then $\delta$ is a closure operator. Hence we can introduce the fuzzy topology $T(\delta)$ on $X$ by $\delta$.

**Proof.** Now we will show that $\delta$ is a closure operator.

1. **(C1)** Suppose that $\mu \neq 0$. There exists $y \in X$ such that $\mu(y) \neq 0$. Consider the fuzzy point $y_\gamma \in I^X$. Here $y_\gamma \land \mu \neq 0$ and therefore $y_\gamma \delta \mu$. Also, $\mu = \bigvee_{\mu(y) \neq 0} y_\gamma$. Hence, $\mu^\delta = \bigvee \{x_\gamma \mid x_\gamma \delta \mu\} \geq \bigvee_{\mu(y) \neq 0} y_\gamma = \mu$. Consequently $\mu^\delta \geq \mu$.

2. **(C2)** For this, it suffices to show that $x_\gamma \delta \mu^\delta$ if and only if $x_\gamma \delta \mu$. Suppose that $x_\gamma \delta \mu$. Then $x_\gamma \delta \mu^\delta$ because of $\mu \leq \mu^\delta$. Conversely, suppose that $x_\gamma \delta \mu^\delta$. Now $y_\gamma \leq \mu^\delta$ implies $y_\gamma \leq \bigvee \{x_\gamma \mid x_\gamma \delta \mu\}$, which gives $y_\gamma \leq x_p$ for some $x_p$ such that $x_p \delta \mu$. We have $y_\gamma \delta \mu$. Thus, we get $x_\gamma \delta \mu^\delta$ and $y_\gamma \delta \mu$ for each $y_\gamma \leq \mu^\delta$. Hence $x_\gamma \delta \mu$.

3. **(C3)**

$$(\mu \lor \rho)^\delta = \bigvee \{x_\gamma \mid x_\gamma \delta (\mu \lor \rho)\}$$

$$= \bigvee \{x_\gamma \mid x_\gamma \delta \mu \text{ or } x_\gamma \delta \rho\}$$

$$= (\bigvee \{x_\gamma \mid x_\gamma \delta \mu\}) \lor (\bigvee \{x_\gamma \mid x_\gamma \delta \rho\})$$

$$= \mu^\delta \lor \rho^\delta$$
(C4) It is also easy to see that $0^\delta = 0$. 

Theorem 4.3. If $(X, \delta)$ is a fuzzy $R$-proximity space, then $T(\delta)$ is fuzzy $R_0$ regular.

Proof. Let $\mu$ be a fuzzy closed set and $x_\gamma$ a fuzzy point such that $x_\gamma \delta \mu$. Then there is a $\rho$ such that $x_\gamma \delta \rho$ and $(1-\rho) \delta \mu$. Hence $x_\gamma \wedge \rho^\delta = 0$ or $x_\gamma \leq 1 - \rho^\delta = \sigma$. On the other hand $\mu \wedge (1-\rho)^\delta = 0$ or $\mu \leq 1 - (1-\rho)^\delta = \lambda$, that is, $1 - \lambda \leq 1 - \mu$. Since $\sigma \wedge \lambda = 0$, there exist fuzzy open sets $\sigma, \lambda$ such that $x_\gamma \leq \sigma \leq 1 - \lambda \leq 1 - \mu$.

To prove that the induced fuzzy topology $T(\delta)$ also satisfies the $R_0$ axiom, i.e., $x_\gamma \in y_\delta$ implies $y_\gamma \in x_\delta$, let $x_\gamma \in y_\delta$. Then $x_\gamma \delta y_\gamma$ if and only if $y_\gamma \delta x_\gamma$ if and only if $y_\gamma \in x_\delta$. 

Theorem 4.4. In a fuzzy $R_0$ regular space $(X, T)$, let $\delta$ be a binary relation on $I^X$ define as follows:

$\mu \delta \rho$ if and only if $\mu^\delta \wedge \rho^\delta \neq 0$,

then $\delta$ is the fuzzy $R$-proximity, which is compatible with $T$.

Proof. We will show that $\delta$ satisfies (FR1)~(FR5).

(FR1) $\mu \delta \rho \implies \mu^\delta \wedge \rho^\delta \neq 0 \implies \rho^\delta \wedge \mu^\delta \neq 0 \implies \rho \delta \mu$.

(FR2)

$$(\mu \vee \rho) \delta \sigma \iff (\mu \vee \rho)^\delta \wedge \sigma^\delta \neq 0 \iff (\mu^\delta \vee \rho^\delta) \wedge \sigma^\delta \neq 0 \iff (\mu^\delta \wedge \sigma^\delta) \vee (\rho^\delta \wedge \sigma^\delta) \neq 0 \iff \mu^\delta \wedge \sigma^\delta \neq 0 \text{ or } \rho^\delta \wedge \sigma^\delta \neq 0 \iff \mu \delta \sigma \text{ or } \rho \delta \sigma.$$  

(FR3)

$$\mu \delta \rho \implies \mu^\delta \wedge \rho^\delta \neq 0 \implies \mu^\delta \neq 0 \text{ and } \rho^\delta \neq 0 \implies \mu \neq 0 \text{ and } \rho \neq 0.$$
Suppose that \( x_\gamma \delta \mu \). Applying the definition of \( \delta \) to \( x_\gamma \delta \mu \) we obtain \( x_\gamma \delta \mu = 0 \) and hence either \( x_\gamma = 0 \) or \( \mu = 0 \). Since \( X \) is regular, there exist fuzzy open sets \( \rho, \sigma \) such that \( x_\gamma \leq \rho \leq 1 - \sigma \leq 1 - \mu \). The following two cases arise:

Cases(1). \( x_\gamma = 0 \). Take \( \sigma = 1 \). Then \( x_\gamma \delta \sigma = 0 \) implies \( x_\gamma \delta \mu \), and

\[
(1 - \sigma) \delta \mu = 0 \text{ implies } (1 - \sigma) \delta \mu.
\]

Cases(2). \( \mu = 0 \). Take \( \sigma = 0 \). Then \( x_\gamma \delta \sigma = 0 \) implies \( x_\gamma \delta \mu \), and

\[
(1 - \sigma) \delta \mu = 0 \text{ implies } (1 - \sigma) \delta \mu.
\]

\[(\text{FR5}) \mu \wedge \rho \neq 0 \implies \mu \delta \rho \neq 0 \implies \mu \delta \rho. \]

THEOREM 4.5. A fuzzy K-proximity space is also R-proximity.

Proof. Let \( (X, \delta) \) be a fuzzy K-proximity space. Then, \( T(\delta) \) is a fuzzy completely regular [4, 8]. Since a completely regular space is a regular, \( T(\delta) \) is a fuzzy R\textsubscript{0} regular. Hence, \((X, \delta)\) is a fuzzy R-proximity space.

References

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