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FUZZY IDEALS AND FUZZY SUBRINGS UNDER TRIANGULAR NORMS

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Abstract. We develop some basic properties of t-fuzzy ideals in a monoid or a group and find the sufficient conditions for a fuzzy set in a division ring to be a t-fuzzy subring and the necessary and sufficient conditions for a fuzzy set in a division ring to be a t-fuzzy ideal.

1. Introduction

The concept of fuzzy sets was first introduced by Zadeh ([8]). Rosenfeld ([4]) used this concept to formulate the notion of fuzzy groups. Since then, many other fuzzy algebraic concepts based on the Rosenfeld's fuzzy groups were developed. Anthony and Sherwood ([1]) redefined fuzzy groups in terms of t-norm which replaced the minimum operation of Rosenfeld's definition. Some properties of these redefined fuzzy groups, which we call t-fuzzy groups in this paper, have been developed by Sherwood ([6]) and Sidky and Mishref ([7]). Sessa ([5]) defined fuzzy ideals with respect to the triangular norms, which we call t-fuzzy ideals in this paper, and developed their properties. As a continuation of these studies, we characterize some basic properties of t-fuzzy ideals and t-fuzzy subrings.

In the section 2 we develop some basic properties of t-fuzzy ideals in a monoid or a group. In the section 3 we find the sufficient conditions for a fuzzy set A in a division ring X to be a t-fuzzy subring without the assumption of $A(u) = 1$, where u is the additive identity element in X, and find the necessary and sufficient conditions for a fuzzy set in a division ring to be a t-fuzzy ideal without the assumption of $A(u) = 1$.

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2. t-fuzzy ideals in a group or a monoid

Definition 2.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in B$, $B(x)$ is called a *membership grade* of x in B.

Anthony and Sherwood ([1]) generalized the definition of a fuzzy groupoid by Rosenfeld ([4]), that is, they replace the stronger condition imposed by the minimum operation with a function $T : [0,1] \times [0,1] \to [0,1]$, called a triangular norm, and they developed some properties of fuzzy groupoids and fuzzy groups.

Definition 2.2. A *t-norm* is a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying, for each p, q, r, s in [0,1],

(1) $T(0,p) = 0,\ T(p,1) = p$
(2) $T(p,q) \leq T(r,s)$ if $p \leq r$ and $q \leq s$
(3) $T(p,q) = T(q,p)$
(4) $T(p, T(q,r)) = T(T(p,q), r))$

Definition 2.3. Let S be a groupoid and T be a t-norm. A function $A : S \to [0,1]$ is a *t-fuzzy groupoid* in S if and only if for every x, y in S, $A(xy) \geq T(A(x), A(y))$. If X is a group, a fuzzy groupoid G is a *t-fuzzy group* in X if and only if for each $x \in X$, $G(x^{-1}) = G(x)$.

For fuzzy sets U, V in a set X, UV has been defined in most articles by

$$(UV)(x) = \begin{cases} \sup\limits_{ab=x} \ \min(U(a), V(b)) & \text{if } ab = x \\ 0 & \text{if } ab \neq x. \end{cases}$$

The following definition by Sessa ([5]) generalizes the above sup-min operation.

Definition 2.4. Let X be a set and let U, V be two fuzzy sets in X. UV is defined by

$$(UV)(x) = \begin{cases} \sup\limits_{ab=x} \ T(U(a), V(b)) & \text{if } ab = x \\ 0 & \text{if } ab \neq x. \end{cases}$$

Proposition 2.5. *Let X be a set and let U, V, and W be fuzzy sets in X. If X is associative, then $(UV)W = U(VW)$.*

Proof. See Proposition 8 in [2]. □

Definition 2.6. Let S be a semigroup and let A, B, C be fuzzy sets in S. The fuzzy set A is a *t-fuzzy left ideal* if and only if $A(xy) \geq A(y)$. The fuzzy set B is called a *t-fuzzy right ideal* if and only if $B(xy) \geq B(x)$. The fuzzy set C is a *t-fuzzy ideal* if and only if $C(xy) \geq \max(C(x), C(y))$.

Proposition 2.7. *Let B be a fuzzy subset in a monoid S. Then*

(1) *SB (or BS) is a t-fuzzy left (or right) ideal of S.*
(2) *SBS is a t-fuzzy ideal of S.*

Proof. (1) Since S is a monoid, $SS = S$. From Proposition 2.5, $(SS)B = S(SB)$ and $B(SS) = (BS)S$. Since $S(x) = 1$, $SB(xy) = ((SS)B)(xy) = (S(SB))(xy) \geq T(S(x), (SB)(y)) = (SB)(y)$. Since $S(y) = 1$, $BS(xy) = (B(SS))(xy) = ((BS)S)(xy) \geq T((BS)(x), S(y)) = BS(x)$.
(2) Since S is a monoid, $SS = S$. From Proposition 2.5, $(SS)BS = S(SBS)$ and $SB(SS) = (SBS)S$. Thus $SBS(xy) = ((SS)BS)(xy) = (S(SBS))(xy) \geq T(S(x), (SBS)(y)) = (SBS)(y)$ and $SBS(xy) = (SB(SS))(xy) = ((SBS)S)(xy) \geq T(SBS(x), S(y)) = (SBS)(x)$. □

Definition 2.8. A fuzzy set in X is called a *fuzzy point* if and only if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is α $(0 < \alpha \leq 1)$, we denote this fuzzy point by x_α, where the point x is called its *support*. The fuzzy point x_α is said to be contained in a fuzzy set A, denoted by $x_\alpha \in A$, if and only if $\alpha \leq A(x)$.

Proposition 2.9. *Let G be a group, let L be a t-fuzzy left ideal of G, let R be a t-fuzzy right ideal of G, and let I be a t-fuzzy ideal.*

(1) *For every fuzzy point g_1 in G, Lg_1 is a t-fuzzy left ideal.*
(2) *For every fuzzy point g_1 in G, g_1R is a t-fuzzy right ideal.*
(3) *If G is an abelian group, $g_1I = Ig_1$ is a t-fuzzy ideal for every fuzzy point g_1 in G.*

Proof. (1) $Lg_1(pq) = \sup_{ab=pq} T(L(a), g_1(b)) = T(L(pqg^{-1}), g_1(g)) = L(pqg^{-1}) \geq L(qg^{-1}) = T(L(qg^{-1}), g_1(g)) = \sup_{cd=q} T(L(c), g_1(d)) = Lg_1(q)$. Thus Lg_1 is a t-fuzzy left ideal.
(2) $g_1R(pq) = \sup_{ab=pq} T(g_1(a), R(b)) = T(g_1(g), R(g^{-1}pq)) = R(g^{-1}pq) \geq R(g^{-1}p) = T(g_1(g), R(g^{-1}p)) = \sup_{cd=p} T(g_1(c), R(d)) = g_1R(p)$. Thus g_1R is a t-fuzzy right ideal.
(3) Since G is an abelian group,

$$\begin{aligned} g_1I(x) &= \sup_{ab=x} T(g_1(a), I(b)) = \sup_{ab=x} T(I(b), g_1(a)) \\ &= \sup_{ba=x} T(I(b), g_1(a)) = Ig_1(x). \end{aligned}$$

That is, $g_1I = Ig_1$. From (1), Ig_1 is a t-fuzzy left ideal. From (2), g_1I is a t-fuzzy right ideal. Thus $g_1I = Ig_1$ is a t-fuzzy ideal. □

3. T-fuzzy subring and t-fuzzy ideals in a ring

DEFINITION 3.1. Let X be a ring with respect to two binary operations $+$ and $\cdot$ and let A be a fuzzy set in X. The fuzzy set A is called a *t-fuzzy subring* of X if A is a t-fuzzy subgroup for $+$ and A is a t-fuzzy subgroupid for the operation $\cdot$ in X.

PROPOSITION 3.2. *Let X be a division ring and let A be a t-fuzzy ring in X. Then $A(e) \geq T(A(x), A(x))$ for $x \neq u$ and $A(u) \geq T(A(x), A(x))$. In particular, $A(u) \geq T(A(e), A(e))$, where u is the additive identity element of X and e is the multiplicative identity element of X.*

Proof. $A(e) = A(x \cdot x^{-1}) \geq T(A(x), A(x^{-1})) = T(A(x), A(x))$ for $x \neq u$. $A(u) = A(x - x) \geq T(A(x), A(-x)) = T(A(x), A(x))$. □

In [5], Sessa shows the necessary and sufficient condition for a fuzzy subset A in a ring X to be a t-fuzzy subring with the assumption of $A(u) = 1$.

THEOREM 3.3. *Let X be a ring and let A be a fuzzy set in X such that $A(u) = 1$, where u is the additive identity element of X. Then A is a t-fuzzy subring of X if and only if $T(A(x), A(y)) \leq min(A(x - y), A(x \cdot y))$ for every $x, y \in X$.*

Proof. See [5, Proposition 2.4]. □

We find the sufficient conditions for a fuzzy subset A in a division ring X to be a t-fuzzy subring without the assumption of $A(u) = 1$.

THEOREM 3.4. *Let X be a division ring and let A be a fuzzy set in X. If $A(x) = A(e)$ for all $x \in X$ with $x \neq u$ and $A(u) \geq T(A(e), A(e))$, then A is a t-fuzzy subring of X, where u is the additive identity element and e is the multiplicative identity element.*

Proof. (i) If $x \neq u$, then $-x \neq u$, and hence $A(x) = A(e) = A(-x)$. If $x = u$, $A(x) = A(-x)$. Thus $A(x) = A(-x)$.
(ii) If $x \neq y$ and $x \neq u$, $A(x - y) = A(e) = A(x) = T(A(x), 1) \geq T(A(x), A(y))$. If $x \neq y$ and $y \neq u$, $A(x - y) = A(e) = A(y) = T(1, A(y)) \geq T(A(x), A(y))$. If $x = y \neq u$, $A(x - y) = A(u) \geq T(A(e), A(e)) = T(A(x), A(y))$. If $x = y = u$, $A(x - y) = A(u) \geq T(1, A(u)) \geq T(A(u), A(u)) = T(A(x), A(y))$. Thus $A(x + y) = A(x - (-y)) \geq T(A(x), A(-y)) = T(A(x), A(y))$.
(iii) If $x = u$, $A(x \cdot y) = A(u) \geq T(A(u), 1) \geq T(A(u), A(y)) = T(A(x), A(y))$. If $y = u$, $A(x \cdot y) = A(u) \geq T(1, A(u)) \geq T(A(x), A(u)) = T(A(x), A(y))$. If $x \neq u$ and $y \neq u$, $A(x \cdot y) = A(e) = A(x) = T(A(x), 1) \geq T(A(x), A(y))$. Thus $A(x \cdot y) \geq T(A(x), A(y))$.
From (i), (ii), and (iii), A is a t-fuzzy subring. □

DEFINITION 3.5. Let X be a ring and let A be a fuzzy set in X. The fuzzy set A is a *t-fuzzy left (or right) ideal* of X if A is a t-fuzzy subring of X and $A(x \cdot y) \geq A(y)$ (or $A(x \cdot y) \geq A(x)$) for every $x, y \in X$. The fuzzy set A is a *t-fuzzy ideal* if A is a t-fuzzy subring of X and $A(x \cdot y) \geq \max(A(x), A(y))$ for every $x, y \in X$.

Liu([3]) showed that a fuzzy subset A of a skew field X is a fuzzy ideal under the operation of minimum iff $A(x) = A(e) \leq A(u)$ for all $x \in X$ such that $x \neq u$, where u is the additive identity element and e is the multiplicative identity element. We generalize this using t-norm operation in the following theorem.

Theorem 3.6. *Let X be a division ring and let A be a fuzzy set in X. Then A is a t-fuzzy ideal of X if and only if $A(u) \geq A(y)$ for all $y \in X$ and $A(x) = A(e)$ for all $x \in X$ with $x \neq u$, where u is the additive identity element and e is the multiplicative identity element.*

Proof. Suppose A is a t-fuzzy ideal. If $y \neq u$, then $u \cdot y = u$, and hence $A(u) = A(u \cdot y) \geq A(y)$. Since X is a division ring, $A(x) = A(x \cdot e) \geq A(e) = A(x \cdot x^{-1}) \geq A(x)$ for $x \in X - \{u\}$. Thus $A(x) = A(e)$ for all $x \in X - \{u\}$.

Suppose $A(u) \geq A(x)$ and $A(x) = A(e)$ for all $x \in X$ with $x \neq u$.
(i) If $x \neq y$ and $x \neq u$, then $A(x - y) = A(e) = A(x) = T(A(x), 1) \geq T(A(x), A(y))$. If $x \neq y$ and $y \neq u$, then $A(x - y) = A(e) = A(y) = T(1, A(y)) \geq T(A(x), A(y))$. If $x = y \neq u$, then $A(x - y) = A(u) \geq A(x) = T(A(x), 1) \geq T(A(x), A(y))$. If $x = y = u$, then $A(x - y) = A(u) = T(A(u), 1) \geq T(A(u), A(u)) \geq T(A(x), A(y))$. Thus $A(x+y) = A(x - (-y)) \geq T(A(x), A(-y)) = T(A(x), A(y))$.
(ii) If $x = u$ and $y \neq u$, then $A(x \cdot y) = A(u) \geq T(A(u), 1) \geq T(A(u), A(e)) = T(A(x), A(y))$. If $y = u$ and $x \neq u$, then $A(x \cdot y) = A(u) \geq T(1, A(u)) \geq T(A(e), A(u)) = T(A(x), A(y))$. If $x = u$ and $y = u$, then $A(x \cdot y) = A(u) = T(1, A(u)) \geq T(A(u), A(u)) = T(A(x), A(y))$. If $x \neq u$ and $y \neq u$, then $A(x \cdot y) = A(e) = A(x) = T(A(x), 1) \geq T(A(x), A(y))$. Thus $A(x \cdot y) \geq T(A(x), A(y))$.
(iii) If $x \neq u$, then $-x \neq u$, and hence $A(x) = A(e) = A(-x)$ for $x \neq u$. If $x = u$, then $x = u = -x$, and hence $A(x) = A(-x)$. Thus $A(-x) = A(x)$.
(iv) If $x \cdot y \neq u$, then $x \neq u$ and $y \neq u$, and hence $A(x \cdot y) = A(e) = A(x) = A(y)$. If $x \cdot y = u$ and $x = y = u$, then $A(x \cdot y) = A(x) = A(y)$. If $x \cdot y = u$, $x = u$, and $y \neq u$, then $A(x \cdot y) = A(u) = A(x) \geq A(y)$. If $x \cdot y = u$, $x \neq u$, and $y = u$, then $A(x \cdot y) = A(u) = A(y) \geq A(x)$. Thus $A(x \cdot y) \geq A(x)$ and $A(x \cdot y) \geq A(y)$.
From (i), (ii), (iii), and (iv), A is a t-fuzzy ideal of X. $\square$

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