FUZZY IDEALS AND FUZZY SUBRINGS UNDER TRIANGULAR NORMS

INHEUNG CHON

Abstract. We develop some basic properties of t-fuzzy ideals in a monoid or a group and find the sufficient conditions for a fuzzy set in a division ring to be a t-fuzzy subring and the necessary and sufficient conditions for a fuzzy set in a division ring to be a t-fuzzy ideal.

1. Introduction

The concept of fuzzy sets was first introduced by Zadeh ([8]). Rosenfeld ([4]) used this concept to formulate the notion of fuzzy groups. Since then, many other fuzzy algebraic concepts based on the Rosenfeld’s fuzzy groups were developed. Anthony and Sherwood ([1]) redefined fuzzy groups in terms of t-norm which replaced the minimum operation of Rosenfeld’s definition. Some properties of these redefined fuzzy groups, which we call t-fuzzy groups in this paper, have been developed by Sherwood ([6]) and Sidky and Mishref ([7]). Sessa ([5]) defined fuzzy ideals with respect to the triangular norms, which we call t-fuzzy ideals in this paper, and developed their properties. As a continuation of these studies, we characterize some basic properties of t-fuzzy ideals and t-fuzzy subrings.

In the section 2 we develop some basic properties of t-fuzzy ideals in a monoid or a group. In the section 3 we find the sufficient conditions for a fuzzy set $A$ in a division ring $X$ to be a t-fuzzy subring without the assumption of $A(u) = 1$, where $u$ is the additive identity element in $X$, and find the necessary and sufficient conditions for a fuzzy set in a division ring to be a t-fuzzy ideal without the assumption of $A(u) = 1$. 

Received July 29, 2002.

2000 Mathematics Subject Classification: 20N25.

Key words and phrases: t-fuzzy subring, t-fuzzy ideal.

This paper was supported by the Natural Science Research Institute of Seoul Women’s University, 2001
2. t-fuzzy ideals in a group or a monoid

DEFINITION 2.1. A function $B$ from a set $X$ to the closed unit interval $[0, 1]$ in $\mathbb{R}$ is called a fuzzy set in $X$. For every $x \in B$, $B(x)$ is called a membership grade of $x$ in $B$.

Anthony and Sherwood ([1]) generalized the definition of a fuzzy groupoid by Rosenfeld ([4]), that is, they replace the stronger condition imposed by the minimum operation with a function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$, called a triangular norm, and they developed some properties of fuzzy groupoids and fuzzy groups.

DEFINITION 2.2. A t-norm is a function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying, for each $p, q, r, s$ in $[0, 1]$,

1. $T(0, p) = 0$, $T(p, 1) = p$
2. $T(p, q) \leq T(r, s)$ if $p \leq r$ and $q \leq s$
3. $T(p, q) = T(q, p)$
4. $T(p, T(q, r)) = T(T(p, q), r))$

DEFINITION 2.3. Let $S$ be a groupoid and $T$ be a t-norm. A function $A : S \rightarrow [0, 1]$ is a t-fuzzy groupoid in $S$ if and only if for every $x, y$ in $S$, $A(xy) \geq T(A(x), A(y))$. If $X$ is a group, a fuzzy groupoid $G$ is a t-fuzzy group in $X$ if and only if for each $x \in X$, $G(x^{-1}) = G(x)$.

For fuzzy sets $U, V$ in a set $X$, $UV$ has been defined in most articles by

$$(UV)(x) = \begin{cases} \sup_{ab=x} \min(U(a), V(b)) & \text{if } ab = x \\ 0 & \text{if } ab \neq x. \end{cases}$$

The following definition by Sessa ([5]) generalizes the above sup-min operation.

DEFINITION 2.4. Let $X$ be a set and let $U, V$ be two fuzzy sets in $X$. $UV$ is defined by

$$(UV)(x) = \begin{cases} \sup_{ab=x} T(U(a), V(b)) & \text{if } ab = x \\ 0 & \text{if } ab \neq x. \end{cases}$$
**Proposition 2.5.** Let $X$ be a set and let $U, V,$ and $W$ be fuzzy sets in $X$. If $X$ is associative, then $(UV)W = U(VW)$.

*Proof.* See Proposition 8 in [2]. □

**Definition 2.6.** Let $S$ be a semigroup and let $A, B, C$ be fuzzy sets in $S$. The fuzzy set $A$ is a $t$-fuzzy left ideal if and only if $A(xy) \geq A(y)$. The fuzzy set $B$ is called a $t$-fuzzy right ideal if and only if $B(xy) \geq B(x)$. The fuzzy set $C$ is a $t$-fuzzy ideal if and only if $C(xy) \geq \max(C(x), C(y))$.

**Proposition 2.7.** Let $B$ be a fuzzy subset in a monoid $S$. Then

1. $SB$ (or $BS$) is a $t$-fuzzy left (or right) ideal of $S$.
2. $SBS$ is a $t$-fuzzy ideal of $S$.

*Proof.*

1. Since $S$ is a monoid, $SS = S$. From Proposition 2.5, $(SS)B = S(SB)$ and $B(SS) = (BS)S$. Since $S(x) = 1$, $SB(xy) = (SS)(xy) = (S(SB))(xy) \geq T(S(x), (SB)(y)) = (SB)(y)$. Since $S(y) = 1$, $BS(xy) = (B(SS))(xy) = (BS)S(xy) \geq T((BS)(x), S(y)) = BS(x)$.

2. Since $S$ is a monoid, $SS = S$. From Proposition 2.5, $(SS)BS = S(SBS)$ and $SB(SS) = (SBS)S$. Thus $SBS(xy) = ((SS)BS)(xy) = (S(SBS))(xy) \geq T(S(x), (SBS)(y)) = (SBS)(y)$ and $SBS(xy) = (SBS)S(xy) \geq T(SBS(x), S(y)) = (SBS)(x)$. □

**Definition 2.8.** A fuzzy set in $X$ is called a fuzzy point if and only if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at $x$ is $\alpha$ ($0 < \alpha \leq 1$), we denote this fuzzy point by $x_\alpha$, where the point $x$ is called its support. The fuzzy point $x_\alpha$ is said to be contained in a fuzzy set $A$, denoted by $x_\alpha \in A$, if and only if $\alpha \leq A(x)$.

**Proposition 2.9.** Let $G$ be a group, let $L$ be a $t$-fuzzy left ideal of $G$, let $R$ be a $t$-fuzzy right ideal of $G$, and let $I$ be a $t$-fuzzy ideal.

1. For every fuzzy point $g_1$ in $G$, $Lg_1$ is a $t$-fuzzy left ideal.
2. For every fuzzy point $g_1$ in $G$, $g_1R$ is a $t$-fuzzy right ideal.
3. If $G$ is an abelian group, $g_1I = Ig_1$ is a $t$-fuzzy ideal for every fuzzy point $g_1$ in $G$. 

Proof. (1) \( Lg_1(pq) = \sup_{ab=pq} T(L(a), g_1(b)) = T(L(pqg^{-1}), g_1(g)) = L(pqg^{-1}) \geq L(qg^{-1}) = T(L(qg^{-1}), g_1(g)) = \sup_{cd=q} T(L(c), g_1(d)) = Lg_1(q) \). Thus \( Lg_1 \) is a t-fuzzy left ideal.

(2) \( g_1R(pq) = \sup_{ab=pq} T(g_1(a), R(b)) = T(g_1(g), R(g^{-1}pq)) = R(g^{-1}pq) \geq R(g^{-1}p) = T(g_1(g), R(g^{-1}p)) = \sup_{cd=p} T(g_1(c), R(d)) = g_1R(p) \). Thus \( g_1R \) is a t-fuzzy right ideal.

(3) Since \( G \) is an abelian group,

\[
g_1I(x) = \sup_{ab=x} T(g_1(a), I(b)) = \sup_{ab=x} T(I(b), g_1(a)) = \sup_{ba=x} T(I(b), g_1(a)) = Ig_1(x).
\]

That is, \( g_1I = Ig_1 \). From (1), \( Ig_1 \) is a t-fuzzy left ideal. From (2), \( g_1I \) is a t-fuzzy right ideal. Thus \( g_1I = Ig_1 \) is a t-fuzzy ideal. \( \square \)

3. T-fuzzy subring and t-fuzzy ideals in a ring

Definition 3.1. Let \( X \) be a ring with respect to two binary operations + and \( \cdot \) and let \( A \) be a fuzzy set in \( X \). The fuzzy set \( A \) is called a t-fuzzy subring of \( X \) if \( A \) is a t-fuzzy subgroup for + and \( A \) is a t-fuzzy subgroup for the operation \( \cdot \) in \( X \).

Proposition 3.2. Let \( X \) be a division ring and let \( A \) be a t-fuzzy ring in \( X \). Then \( A(e) \geq T(A(x), A(x)) \) for \( x \neq u \) and \( A(u) \geq T(A(x), A(x)) \). In particular, \( A(u) \geq T(A(e), A(e)) \), where \( u \) is the additive identity element of \( X \) and \( e \) is the multiplicative identity element of \( X \).

Proof. \( A(e) = A(x \cdot x^{-1}) \geq T(A(x), A(x^{-1})) = T(A(x), A(x)) \) for \( x \neq u \). \( A(u) = A(x - x) \geq T(A(x), A(-x)) = T(A(x), A(x)) \). \( \square \)

In [5], Sessa shows the necessary and sufficient condition for a fuzzy subset \( A \) in a ring \( X \) to be a t-fuzzy subring with the assumption of \( A(u) = 1 \).
Theorem 3.3. Let $X$ be a ring and let $A$ be a fuzzy set in $X$ such that $A(u) = 1$, where $u$ is the additive identity element of $X$. Then $A$ is a t-fuzzy subring of $X$ if and only if $T(A(x), A(y)) \leq \min(A(x - y), A(x \cdot y))$ for every $x, y \in X$.

Proof. See [5, Proposition 2.4].

We find the sufficient conditions for a fuzzy subset $A$ in a division ring $X$ to be a t-fuzzy subring without the assumption of $A(u) = 1$.

Theorem 3.4. Let $X$ be a division ring and let $A$ be a fuzzy set in $X$. If $A(x) = A(e)$ for all $x \in X$ with $x \neq u$ and $A(u) \geq T(A(e), A(e))$, then $A$ is a t-fuzzy subring of $X$, where $u$ is the additive identity element and $e$ is the multiplicative identity element.

Proof. (i) If $x \neq u$, then $-x \neq u$, and hence $A(x) = A(e) = A(-x)$. If $x = u$, $A(x) = A(-x)$. Thus $A(x) = A(-x)$.

(ii) If $x \neq y$ and $x \neq u$, $A(x - y) = A(e) = A(x) = T(A(x), 1) \geq T(A(x), A(y))$. If $x \neq y$ and $y \neq u$, $A(x - y) = A(e) = T(1, A(y)) \geq T(A(x), A(y))$. If $x = y \neq u$, $A(x - y) = A(u) \geq T(A(e), A(e)) = T(A(x), A(y))$. If $x = y$, $A(x - y) = A(u) \geq T(1, A(u)) \geq T(A(u), A(u)) = T(A(x), A(y))$. Thus $A(x + y) = A(x - (-y)) \geq T(A(x), A(-y)) = T(A(x), A(y))$.

(iii) If $x = u$, $A(x \cdot y) = A(u) \geq T(A(u), 1) \geq T(1, A(u)) = T(A(x), A(y))$. If $y = u$, $A(x \cdot y) = A(u) \geq T(1, A(u)) \geq T(A(x), A(u)) = T(A(x), A(y))$. If $x \neq u$ and $y \neq u$, $A(x \cdot y) = A(e) = A(x) = T(A(x), 1) \geq T(A(x), A(y))$. Thus $A(x \cdot y) \geq T(A(x), A(y))$.

From (i), (ii), and (iii), $A$ is a t-fuzzy subring.

Definition 3.5. Let $X$ be a ring and let $A$ be a fuzzy set in $X$. The fuzzy set $A$ is a t-fuzzy left (or right) ideal of $X$ if $A$ is a t-fuzzy subring of $X$ and $A(x - y) \geq A(y)$ (or $A(x - y) \geq A(x)$) for every $x, y \in X$. The fuzzy set $A$ is a t-fuzzy ideal if $A$ is a t-fuzzy subring of $X$ and $A(x \cdot y) \geq \max(A(x), A(y))$ for every $x, y \in X$.

Liu[3] showed that a fuzzy subset $A$ of a skew field $X$ is a fuzzy ideal under the operation of minimum if $A(x) = A(e) \leq A(u)$ for all $x \in X$ such that $x \neq u$, where $u$ is the additive identity element and $e$ is the multiplicative identity element. We generalize this using t-norm operation in the following theorem.
Theorem 3.6. Let $X$ be a division ring and let $A$ be a fuzzy set in $X$. Then $A$ is a $t$-fuzzy ideal of $X$ if and only if $A(u) \geq A(y)$ for all $y \in X$ and $A(x) = A(e)$ for all $x \in X$ with $x \neq u$, where $u$ is the additive identity element and $e$ is the multiplicative identity element.

Proof. Suppose $A$ is a $t$-fuzzy ideal. If $y \neq u$, then $u \cdot y = u$, and hence $A(u) = A(u \cdot y) \geq A(y)$. Since $X$ is a division ring, $A(x) = A(x \cdot e) \geq A(e) = A(x \cdot x^{-1}) \geq A(x)$ for $x \in X - \{u\}$. Thus $A(x) = A(e)$ for all $x \in X - \{u\}$.

Suppose $A(u) \geq A(x)$ and $A(x) = A(e)$ for all $x \in X$ with $x \neq u$.

(i) If $x \neq y$ and $x \neq u$, then $A(x - y) = A(e) = A(x) = T(A(x), 1) \geq T(A(x), A(y))$. If $x \neq y$ and $y \neq u$, then $A(x - y) = A(e) = A(y) = T(1, A(y)) \geq T(A(x), A(y))$. If $x = y \neq u$, then $A(x - y) = A(u) = T(A(u), 1) \geq T(A(u), A(u)) \geq T(A(x), A(y))$. If $x = y = u$, then $A(x - y) = A(u) = T(A(u), 1) \geq T(A(u), A(u)) \geq T(A(x), A(y))$. Thus $A(x + y) = A(x) = T(A(x), A(−y)) = T(A(x), A(y))$.

(ii) If $x = u$ and $y \neq u$, then $A(x \cdot y) = A(u) \geq T(A(u), 1) \geq T(A(u), A(e)) = T(A(x), A(y))$. If $y = u$ and $x \neq u$, then $A(x \cdot y) = A(u) \geq T(1, A(u)) \geq T(A(e), A(u)) = T(A(x), A(y))$. If $x = u$ and $y = u$, then $A(x \cdot y) = A(u) = T(1, A(u)) \geq T(A(u), A(u)) = T(A(x), A(y))$. If $x \neq u$ and $y \neq u$, then $A(x \cdot y) = A(e) = A(x) = T(A(x), 1) \geq T(A(x), A(y))$. Thus $A(x \cdot y) \geq T(A(x), A(y))$.

(iii) If $x \neq u$, then $−x \neq u$, and hence $A(x) = A(e) = A(−x)$ for $x \neq u$. If $x = u$, then $x = u = −x$, and hence $A(x) = A(−x)$. Thus $A(−x) = A(x)$.

(iv) If $x \cdot y \neq u$, then $x \neq u$ and $y \neq u$, and hence $A(x \cdot y) = A(e) = A(x) = A(y)$. If $x \cdot y = u$ and $x = y = u$, then $A(x \cdot y) = A(x) = A(y)$. If $x \cdot y = u$, $x \neq u$, and $y = u$, then $A(x \cdot y) = A(u) = A(y) \geq A(x)$. Thus $A(x \cdot y) \geq A(x)$ and $A(x \cdot y) \geq A(y)$.

From (i), (ii), (iii), and (iv), $A$ is a $t$-fuzzy ideal of $X$. □

References


Department of Mathematics
Seoul Women’s University
Seoul 139-774, Korea