Kangweon-Kyungki Math. Jour. **10** (2002), No. 2, pp. 167–171

ON A FRAME IN THE SPACE $L^2(\mathbb{R})$

Phil Ung Chung* and Song Ho Han

1. Introduction

Frames were introduced by Duffin and Schaeffer(1952), in the context of non-harmonic Fourier series and they were also reviewed by Young(1980). Here we shall review their definition, given in [1]:

Definition. *A family of functions* $(\varphi_j)_{j\in\mathbb{Z}}$ *in a Hilbert space* $\mathcal{H}$ *is called a frame if there exist* A *and* B $(0 < A \le B < \infty)$ *so that, for all* $f \in \mathcal{H}$,

$$A\|f\| \le \sum |< f, \varphi_j >|^2 \le B\|f\|.$$

We call A *and* B *the frame bounds.*

In $L^2(\mathbb{R})$, a Weyl-Heisenberg frame(WH-frame), given by

$$\left\{g_{m,n}(x) \stackrel{\text{def}}{=} g(x-nq_0)e^{2\pi imp_0x} | n, m \in \mathbb{Z}\right\}$$

and an Affine frame, given by

$$\left\{h_{m,n}(x) \stackrel{\text{def}}{=} a_0^{-\frac{m}{2}} h(a_0^{-m}(x-a_0^m nb_0)) | n, m \in \mathbb{Z}\right\}$$

are commonly used as the analysis tools for the signal process in time-frequency domain, where p_0, q_0, a_0 and b_0 are the appropriately chosen positive constants, and g, h are the functions in $L^2(\mathbb{R})$. A WH-frame could be used to choose the data for a given signal at equidistant points

Received August 21, 2002.
2000 Mathematics Subject Classification: 65T99,68M10.
Key words and phrases: Fourier transform, frame.
*This paper was accomplished with Research Fund provided by Kangwon National University, Supprot for 2000 faculty Research Abroad

in the domain, but it doesn't have a high resolution, which means that translation parts nq_0 do not depend on the frequency mp_0. On the other hand an Affine frame have a high resolution which means that translation parts $a_0^m nb_0$ are reciprocally proportional to the frequency a_0^{-m} ,but its intervals for choosing data $a_0^{-m}b_0$ aren't equidistant.

In this paper, we shall construct another new frame which may be used to choose data for a signal at equidistant points of the domain and, at a same time, may have a high resolution.

Throughout the present paper, the symbols $\hat{f}$, $f^{\vee}$, and $\bar{f}$ denote the Fourier transform, the inverse Fourier transform, and complex conjugate of $f \in L^2(\mathbb{R})$ respectively.

2. Another new frame in $L^2(\mathbb{R})$

In this section we shall construct another frame which may be used to choose data at equidistant points of the domain and, at the same time, may have a high resolution.

THEOREM 1. *Let $g \in L^2(\mathbb{R})$ such that $\|g\| \neq 0$ and supp $\hat{g}(\omega) \subset [a, a+\frac{1}{q_0}], q_0 > 0$. Then, for any positive real constant p_0 so that*

$$H(\omega) \stackrel{\text{def}}{=} \frac{1}{q_0} \sum_{m\in\mathbb{Z}} |\hat{g}(\omega - mp_0)|^2$$

may have a positive lower bound A and an upper bound $B(< \infty)$, the family of functions

$$g_{m,n}(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{|\lambda(m)|}} g(x - \frac{nq_0}{\lambda(m)}) e^{2\pi i m p_0 x}, \quad m, n \in \mathbb{Z} \tag{2}$$

constitutes a frame in $L^2(\mathbb{R})$, where $\lambda(m)$ is an arbitrary real valued function of integers such that $1 \leq |\lambda(m)| < \infty$.

proof. Since $[f(x)e^{2\pi ipx}]^\wedge(\omega) = \hat{f}(\omega - p)$, we have, for any $f \in$

$L^2(\mathbb{R})$,
(3)
$$\sum_{m,n\in\mathbb{Z}} |< f, g_{m,n} >|^2$$
$$= \sum_{m,n} |< \hat{f}(\omega), \frac{1}{\sqrt{|\lambda(m)|}}\hat{g}(\omega - mp_0)e^{-2\pi i \frac{nq_0}{\lambda(m)}(\omega - mp_0)} >|^2$$
$$= \sum_{m,n} \frac{1}{|\lambda(m)|} |\int_{\mathbb{R}} \bar{\hat{f}}(\omega + mp_0)\hat{g}(\omega)e^{-2\pi i \frac{nq_0}{\lambda(m)}\omega} d\omega|^2$$
$$= \sum_{m,n} \frac{1}{|\lambda(m)|} |\int_a^{a+\frac{1}{q_0}} \bar{\hat{f}}(\omega + mp_0)\hat{g}(\omega)e^{-2\pi i \frac{nq_0}{\lambda(m)}\omega} d\omega|^2$$
$$= \sum_{m} \frac{1}{|\lambda(m)|} \sum_n |\int_a^{a+\frac{|\lambda(m)|}{q_0}} \bar{\hat{f}}(\omega + mp_0)\hat{g}(\omega)e^{-2\pi i \frac{nq_0}{|\lambda(m)|}\omega} d\omega|^2,$$
while the integral
$$\frac{q_0}{|\lambda(m)|}\int_a^{a+\frac{|\lambda(m)|}{q_0}} \bar{\hat{f}}(\omega + mp_0)\hat{g}(\omega)e^{-2\pi i \frac{nq_0}{|\lambda(m)|}\omega} d\omega$$
is the Fourier coefficient of $\bar{\hat{f}}(\omega + mp_0)\hat{g}(\omega)$. Hence we obtain
(4)
$$\sum_{m,n} |< f, g_{m,n} >|^2 = \frac{1}{q_0}\sum_m \int_a^{a+\frac{|\lambda(m)|}{q_0}} |\hat{f}(\omega + mp_0)\hat{g}(\omega)|^2 d\omega$$
$$= \frac{1}{q_0}\sum_m \int_{a+mp_0}^{a+mp_0+\frac{|\lambda(m)|}{q_0}} |\hat{f}(\omega)\hat{g}(\omega - mp_0)|^2 d\omega$$
$$= \int_{\mathbb{R}} |\hat{f}(\omega)|^2 H(\omega) d\omega,$$
which proves our assertion by virtue of our hypothesis. □

It is noteworthy that the frame constructed in the above theorem may be used to choose data equidistantly and also it has a high resolution in the frequency domain.

With the same notations as in the Theorem 1, we have the following corollaries:

Corollary 1. *If $H(\omega) = 1$ and $\|g_{m,n}\| = 1$, then $\{g_{m,n} | m, n \in \mathbb{Z}\}$ constitutes an orthonormal frame in $L^2(\mathbb{R})$.*

proof. Assume $H(\omega) = 1$. Then we have, for any fixed $k, l \in \mathbb{Z}$,

$$\sum_{m,n} | < g_{k,l}, g_{m,n} > |^2 = | < g_{k,l}, g_{k,l} > |^2$$

by virtue of (4). On the other hand we also have

$$\sum_{m,n} | < g_{k,l}, g_{m,n} > |^2 = | < g_{k,l}, g_{k,l} > |^2 + \sum_{(m,n) \neq (k,l)} | < g_{k,l}, g_{m,n} > |^2.$$

Hence we obtain $< g_{k,l}, g_{m,n} >= 0$ for $(m,n) \neq (k,l)$. $\square$

Corollary 2. *Let*

$$\hat{\phi}(\omega) \stackrel{\text{def}}{=} \frac{\sqrt{q_0}\hat{g}(\omega)}{\{\sum_m |\hat{g}(\omega + mp_0)|^2\}^{\frac{1}{2}}},$$

then the family of functions

$$\{\frac{1}{\sqrt{|\lambda(m)|}} \phi(x - \frac{nq_0}{\lambda(m)}) e^{2\pi i m p_0 x} | m, n \in \mathbb{Z}\}$$

constitutes an orthogonal frame in $L^2(\mathbb{R})$.

proof. ϕ clearly satisfies the conditions given in Theorem 1, and $\frac{1}{q_0} \sum_m |\hat{\phi}(\omega - mp_0)|^2 = 1$. Hence our assertion follows from Theorem 1 and Corollary 1. $\square$

Corollary 3. *Let $\hat{g}(\omega)$ is continuous and supp $\hat{g}(\omega) \subset [a, b]$ and let p_0 be any positive constant such that $p_0 < L = b - a$, then $\{g_{m,n} | m, n \in \mathbb{Z}\}$ constitutes a frame in $L^2(\mathbb{R})$.*

proof. Since $H(\omega)$ is a periodic function with period p_0, Let us choose the interval $[a - \frac{p_0}{2}, b + \frac{p_0}{2}]$ so that it may contain at least one period of $\hat{\phi}$. Then $H(\omega)$ also has positive lower bound and upper bound on the chosen interval. Hence $H(\omega)$ has a positive lower bound and upper bound on $\mathbb{R}$, so that $\{g_{m,n} | m, n \in \mathbb{Z}\}$ constitutes a frame in $L^2(\mathbb{R})$ by virtue of Theorem 1. $\square$

COROLLARY 4. *If the orthonormal frame* $\{g_{m,n}\}$ *spans the space* $L^2(\mathbb{R})$, *then we have, for any* $f \in L^2(\mathbb{R})$,

$$f = \sum_{m,n\in\mathbb{Z}} < f, g_{m,n} > g_{m,n}. \tag{5}$$

proof. By virtue of (4), we have , for any $f \in L^2(\mathbb{R})$,

$$\sum_{m,n} | < f, g_{m,n} > |^2 =< f, f >,$$

which implies, by the polarization identity,

$$< f, g >= \sum_{m,n} < f, g_{m,n} >< g_{m,n}, g > .$$

Hence we proved our assertion. □

References

1. C. Gasquet and P. Witomski, *Fourier Analysis and Applications*, Springer, 1999.
2. C.K. Chui, *An Introduction to Wavelets*, Academic Press, New York, 1992.
3. Cui Mingen and et al, *Fourier Trnasform and Wavelet Analysis*, Kyung Moon, 2001.
4. Ingrid Daubeachies, *Ten Lectures On Wavelets*, CBMMS-NSF, 1992.

Phil Ung Chung
Dept. of Math.
Kangwon National University
Chunchon 200-701, KOREA
E-mail: puchung@kangwon.ac.kr

Song Ho Han
Dept. of Math.
Kangwon National University
Chunchon 200-701, KOREA
E-mail: hansong@kangwon.ac.kr