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FLOWS OF CHARACTERISTIC 0 AND REGULARITIES

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Abstract. The purpose of this paper is to study and characterize the flows of characteristic 0. It is shown that the homomorphic image of distal flow of characteristic 0 is a distal flow of characteristic 0. It is also shown that the closure of every orbit in a 0-graphic flow is regular minimal.

1. Introduction and preliminaries

S. Elaydi and S.K. Kaul [2] have introduced the flow of characteristic 0 which is more generalized than the minimal flow. In this paper we study and characterize the flows of characteristic 0.

Throughout this paper, we assume that the phase spaces of the flows are compact Hausdorff spaces.

For a flow (X,T), we define the first prolongation set $D(x)$ of x in X by

$$D(x) = \{y \in X \mid \; x_i t_i \to y \text{ for some } x_i \to x, \; t_i \in T\}.$$

It is easily checked that $y \in D(x)$ if and only if $x \in D(y)$.

A point $x \in X$ is said to be of characteristic 0 under T if $D(x) = \overline{xT}$. The flow (X,T) is said to be of characteristic 0 under T if every point in X is of characteristic 0 under T.

Lemma 1.1. *Let (X,T) be a flow and let $x \in X$. Then $y \in D(x)$ if and only if there are nets $\{x_i\}$ in X and $\{t_i\}$ in T such that $\lim x_i = x$ and $\lim x_i t_i = y$.*

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Lemma 1.2. [**?**]. *Let (X,T) be a flow. Then the following statements are pairwise equivalent.*

(1) (X,T) is distal.

(2) (X^I,T) is pointwise almost periodic where I is a set with at least two elements.

(3) $(X\times X,T)$ is pointwise almost periodic.

Lemma 1.3. [**?**] *Let $\varphi : (X,T) \longrightarrow (Y,T)$ be an epimorphism and $\check{\varphi} : X\times X \longrightarrow Y\times Y$ the induced by φ. Then:*

(1) $\check{\varphi}\ P(X,T) \subset P(Y,T)$.

(2) If (Y,T) is pointwise almost periodic, $\check{\varphi}\ P(X,T) = P(Y,T)$.

2. Some properties of the flows of characteristic 0

Theorem 2.1. [**?**] *The following statements are pairwise equivalent.*

(1) The flow (X,T) is of characteristic 0.

(2) The relation $A = \{(x,y) \mid y \in \overline{xT}\ \}$ is a closed invariant equivalence relation on X.

Corollary 2.2. *If (X,T) is almost periodic, then (X,T) is of characteristic 0.*

Proof. Note that if the flow (X,T) is almost periodic, then
$A = \{(x,y) \mid y \in \overline{xT}\}$ is a closed invariant equivalence relation on X (see Proposition 4.10 in [3]). □

Corollary 2.3. *The following statements are pairwise equivalent.*

(1) The flow (X,T) is almost periodic.

(2) The flow $(X\times X,T)$ is of characteristic 0.

Proof. It follows from Proposition [3; 4.11] that if the flow (X,T) is almost periodic if and only if the orbit closure relation on $(X\times X,T)$ is a closed invariant equivalence relation on $X\times X$. By Theorem 2.1 this implies that the flow $(X\times X,T)$ is of characteristic 0. □

Corollary 2.4. *If $(X\times X,T)$ is of characteristic 0, then (X,T) is of characteristic 0.*

Remark 2.5. The property of characteristic 0 is not preserved under the product operation. There exists a flow (X,T) which is of characteristic 0 while the induced square flow $(X\times X,T)$ fails to be of characteristic 0 (see Example 5.3 in [2]).

LEMMA 2.6. *If (X,T) is of characteristic 0, then (X,T) is pointwise almost periodic.*

Proof. Assume that (X,T) is of characteristic 0. To show that the flow (X,T) is pointwise almost periodic it suffices to show that $\overline{xT}$ is a compact minimal subset of X $(x \in X)$ by Proposition [3; 2.5]. Since (X,T) is of characteristic 0, we have that if $y \in \overline{xT}$, then $y \in D(x)$. It follows that $x \in D(y) = \overline{yT}$. This implies that $\overline{xT}$ is minimal. Recall that a closed subset of a compact space is compact. The proof is now completed. □

REMARK 2.7. In a flow (X,T) the concepts of Elaydi-Kaul regularity (that is, for any $x \in X$, any open set U in X and any subset P of T with $\overline{xP} \subset U$, there exists an open neighborhood V of X such that $VP \subset U$) and almost periodicity coincide in flows when the phase space is locally compact Hausdorff and the closure of every orbit is compact (see Remark 4.6 in [2]). If the flow with a compact Hausdorff phase space is Elaydi-Kaul regular, then it is easily seen that it is of characteristic 0 by Corollary 2.2.

LEMMA 2.8. *Let $\varphi : (X,T) \longrightarrow (Y,T)$ be a homomorphism. Then $\varphi(D(x)) \subset D(\varphi(x))$ for each $x \in X$.*

Proof. Suppose $y \in \varphi(D(x))$. Then there exists $x' \in D(x)$ for which $y = \varphi(x')$. Hence there are nets $\{x_i\}$ in X and $\{t_i\}$ in T such that $\lim x_i = x$ and $\lim x_i t_i = x'$. Since φ is a homomorphism, it follows that $y = \varphi(\lim x_i t_i) = \lim(\varphi(x_i t_i)) = \lim \varphi(x_i) t_i$ and $\lim(\varphi(x_i)) = \varphi(x)$. This proves that $y \in D(\varphi(x))$. □

THEOREM 2.9. *Let $\varphi : (X,T) \longrightarrow (Y,T)$ be an epimorphism and let (X,T) be distal. Then $\varphi(D(x)) = D(\varphi(x))$ for each $x \in X$.*

Proof. In order to show that $\varphi(D(x)) = D(\varphi(x))$ for each $x \in X$, it is sufficient to show that $D(\varphi(x)) \subset \varphi(D(x))$ by Lemma 2.8. Let $y \in D(\varphi(x))$. Then there are nets $\{y_i\}$ in Y and $\{t_i\}$ in T such that $\lim y_i = \varphi(x)$ and $\lim y_i t_i = y$. Because φ is an epimorphism, there exists a net $\{x_i\}$ in X for which $\varphi(x_i) = y_i$. Since the phase space X is compact, we have a convergent subnet $\{x_{i_\alpha}\}$ of $\{x_i\}$ such that $\lim x_{i_\alpha} = x_0$ for some $x_0 \in X$. This implies that $\lim x_{i_\alpha} t_{i_\alpha} \in D(x_0)$ and $\varphi(\lim x_{i_\alpha} t_{i_\alpha}) = \lim \varphi(x_{i_\alpha}) t_{i_\alpha} = \lim y_{i_\alpha} t_{i_\alpha}$. But since Y is Hausdorff, we have that $\lim y_{i_\alpha} t_{i_\alpha} = y$. Thus $y \in \varphi(D(x_0))$. Since $\varphi(x_0) = \varphi(\lim x_{i_\alpha}) = \lim y_{i_\alpha} = \varphi(x)$, it follows that $\check{\varphi}(x, x_0) = (\varphi(x), \varphi(x_0)) \in P(Y,T)$. Since

(X,T) is distal, it follows from Corollary [3; 5.7] that (Y,T) is distal and hence it is pointwise almost periodic. Thus $\check{\varphi}\ P(X,T) = P(Y,T)$ by Lemma 1.3. This means that $(x, x_0) \in P(X,T)$. Moreover, (x, x_0) is an almost periodic point by Lemma 1.2. Hence $x = x_0$, which means that $y \in \varphi(D(x))$ (see Remark 5.11.4 in [3]). □

COROLLARY 2.10. *Let $\varphi : (X,T) \longrightarrow (Y,T)$ be an epimorphism and let (X,T) be a distal flow of characteristic 0. Then (Y,T) is also a distal flow of characteristic 0.*

Proof. Since (X,T) is distal, we have from Corollary [3; 5.7] that (Y,T) is distal. In order to show that (Y,T) is of characteristic 0, suppose $y \in Y$. Then there exists a $x \in X$ for which $y = \varphi(x)$. Corollary 2.10 now follows the fact that $D(y) = D(\varphi(x)) = \varphi(D(x)) = \varphi(\overline{xT}) = \overline{\varphi(x)T} = \overline{yT}$. □

Y. K. Kim [5] has defined the 0-graphic flow as follows.

DEFINITION 1. A 0-graphic flow is a flow (X,T) of characteristic 0 such that any minimal subset of $(X \times X, T)$ is contained in a graph Γ_t for some $t \in T$, where T is abelian.

LEMMA 2.11. [?] *Let (X,T) be of characteristic 0 with T abelian. Then the following are pairwise equivalent.*

(1) The flow (X,T) is 0-graphic.

(2) If $x, y \in X$, then there is a $t \in T$ such that $(xt, y) \in P(X,T)$.

(3) For any almost periodic point (x,y) of $(X \times X, T)$, there exists $t \in T$ such that $y = xt = \pi^t(x)$.

THEOREM 2.12. *Let $\varphi : (X,T) \longrightarrow (Y,T)$ be an epimorphism of flows of characteristic 0. Then the following statement is valid.*

(X,T) is 0-graphic if and only if (Y,T) is 0-graphic.

Proof. Assume that (X,T) is 0-graphic. To show that (Y,T) is 0-graphic, suppose $y_1, y_2 \in Y$. Then there exist $x_1, x_2 \in X$ for which $y_1 = \varphi(x_1),\ y_2 = \varphi(x_2)$. Since X is 0-graphic, it follows that there is a $t \in T$ such that $(x_1 t, x_2) \in P(X,T)$. Then, according to Lemma 1.3 and Lemma 2.6, $\check{\varphi}\ P(X,T) = P(Y,T)$. This implies that $\check{\varphi}(x_1 t, x_2) = (\varphi(x_1)t, \varphi(x_2)) = (y_1 t, y_2) \in P(Y,T)$. This completes the proof of this part.

Conversely, assume that (Y,T) is 0-graphic. Let (x, x') be an almost periodic point of $(X \times X, T)$ and let $y = \varphi(x), y' = \varphi(x')$. Then $(y, y') =$

$\check{\varphi}(x, x')$ is also an almost periodic point of $(Y \times Y, T)$. Since Y is 0-graphic, it follows that there is a $t \in T$ such that $y' = yt$ and therefore $\varphi(x') = \varphi(xt)$. Thus $\check{\varphi}(xt, x') \in P(Y, T)$. Note that by Lemma 2.6 and Lemma 1.3 $\check{\varphi}\ P(X, T) = P(Y, T)$. This implies that $(xt, x') \in P(X, T)$. But since T is abelian, we have that $\pi^t \times \imath_d$ is a homomorphism. Then $\pi^t \times \imath_d(x, x') = (xt, x')$ and hence (xt, x') is an almost periodic point of $(X \times X, T)$. Hence $xt = x'$. This proves that (X, T) is 0-graphic by Lemma 2.11.3. □

THEOREM 2.13. [?] *A 0-graphic flow is regular.*

Proof. Let (X, T) be 0-graphic. Let M be a minimal subset of (X, T), and let $x, y \in X$. Then there is a $t \in T$ such that $(xt, y) \in P(X, T)$ by Lemma 2.11. Since T is abelian, it follows that $t : x \mapsto xt$ is an automorphism of X onto X. Since M is invariant, we have that $t\,|_M$ is an endomorphism of M into M. Hence (X, T) is regular. □

COROLLARY 2.14. *If (X, T) is 0-graphic, then the closure of every orbit is regular minimal.*

Proof. This follows from Lemma 2.6, Theorem 2.13 and the fact that x is an almost periodic point if and only if $\overline{xT}$ is a compact minimal subset of X $(x \in X)$. □

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