TRANSLATION THEOREM ON FUNCTION SPACE

JAE GIL CHOI AND YOUNG SEO PARK

Abstract. In this paper, we use a generalized Brownian motion process to define a translation theorem. First we establish the translation theorem for function space integrals. We then obtain the general translation theorem for functionals on function space.

1. Introduction.

In [1], Cameron and Martin introduced the transformation of Wiener integrals under the translation. In [4], Chang, Skoug and Park studied translation theorems for Fourier-Feynman transforms and conditional Fourier-Feynman transforms.

In this paper, we study a translation theorem for functionals on function space but with $x$ in a very general function space $C_{a,b}[0,T]$ rather than in the Wiener space. The Wiener process used in [1,4] is free of drift and is stationary in time while the stochastic processes used in this paper is nonstationary in time and is subject to a drift $a(t)$.

In Section 2 of this paper, we give the basic concepts and notations. In Section 3, we study a translation theorem for function space integrals. Finally, in Section 4, we establish the general translation theorem for functionals on function space.

2. Definitions and preliminaries.

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real valued stochastic process $Y$ on $(\Omega, \mathcal{B}, P)$ and $D$ is called a generalized Brownian motion process if $Y(0, \omega) = 0$ almost everywhere and
for $0 = t_0 < t_1 < \cdots < t_n \leq T$, the $n$-dimensional random vector $(Y(t_1, \omega), \cdots, Y(t_n, \omega))$ is normally distributed with density function

$$K(\vec{t}, \vec{\eta}) = ((2\pi)^n \prod_{j=1}^{n} (b(t_j) - b(t_{j-1})))^{-1/2}$$

\begin{equation}
\cdot \exp\left\{-\frac{1}{2} \sum_{j=1}^{n} \frac{(\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1}))^2}{b(t_j) - b(t_{j-1})}\right\}
\end{equation}

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \cdots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0, T]$ with $a(0) = 0$, $a'(t) \in L^2[0, T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0, T]$.

As explained in [8, p.18-20], $Y$ induces a probability measure $\mu$ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest $\sigma$-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process $Y$ determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process $Y$ determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s, t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [8, p.187], the probability measure $\mu$ induced by $Y$, taking a separable version, is supported by $C_{a,b}[0, T]$ (which is equivalent to the Banach space of continuous functions $x$ on $[0, T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0, T], \mathcal{B}(C_{a,b}[0, T]), \mu)$ is the function space induced by $Y$ where $\mathcal{B}(C_{a,b}[0, T])$ is the Borel $\sigma$-algebra of $C_{a,b}[0, T]$.

Let $L^2_{a,b}[0, T]$ be the Hilbert space of functions on $[0, T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0, T]$ induced by $a(\cdot)$ and $b(\cdot)$: i.e.,

\begin{equation}
L^2_{a,b}[0, T] = \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\}
\end{equation}

where $|a|(t)$ denotes the total variation of the function $a$ on the interval $[0, t]$. 
Translation theorem on function space 19

For convenience, let $BV[0, T]$ be the space of bounded variation functions on $[0, T]$. We denote the function space integral of a $B(C_{a,b}[0, T])$-measurable functional $F$ by

$$\int_{C_{a,b}[0, T]} F(x) d\mu(x)$$

whenever the integral exists.

3. Translation theorem for function space integrals.

Let $(C_{a,b}[0, T], B(C_{a,b}[0, T]), \mu)$ be the function space induced by the generalized Brownian motion process defined in Section 1. In this section we will obtain a translation theorem for function space integrals over $(C_{a,b}[0, T], B(C_{a,b}[0, T]), \mu)$.

For a partition $\tau = \{t_1, \cdots, t_n\}$ of $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$, define a function $X_\tau : C_{a,b}[0, T] \to \mathbb{R}$ by

$$X_\tau(x) = (x(t_1), \cdots, x(t_n)).$$

For $x \in C_{a,b}[0, T]$, define the function $[X_\tau(x)] \equiv [x]_n : [0, T] \to \mathbb{R}$ by

$$(3.1)\ [x]_n(t) = x(t_{j-1}) + \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} (x(t_j) - x(t_{j-1}))$$

for each $t \in [t_{j-1}, t_j], j = 1, 2, \cdots, n$. In case, $[x]_n$ is called the polygonalized form of $x$. Similarly, for $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$, define the function $[\xi]_n : [0, T] \to \mathbb{R}$ by

$$(3.2)\ [\xi]_n(t) = \xi_{j-1} + \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} (\xi_j - \xi_{j-1})$$

for each $t \in [t_{j-1}, t_j], j = 1, 2, \cdots, n$, and $\xi_0 = 0$.

For any positive integer $n$, define the point $t_j$ by

$$t_j = \frac{T}{n} j$$

where $j = 1, 2, \cdots, n$. 

Lemma 3.1. Let $[x]_n$ be the polygonalized form of $x$ as in (3.1) and let $F$ be a bounded and continuous functional on $C_{a,b}[0,T]$. Then

$$
\lim_{n \to \infty} \int_{C_{a,b}[0,T]} F([x]_n) d\mu(x) = \int_{C_{a,b}[0,T]} F(x) d\mu(x).
$$

Proof. Let $\varepsilon > 0$ be given. For any $x \in C_{a,b}[0,T]$, there exists an integer $n_0 = n_0(\varepsilon)$ such that for all $|t' - t''| \leq 1/n_0$, we have

$$
|x(t') - x(t'')| < \frac{\varepsilon}{2}.
$$

By using (3.1) and (3.4), we have for each $n \geq n_0$ and $t_{j-1} \leq t \leq t_j$,

$$
|x(t) - [x]_n(t)| = \left| x(t) - x(t_{j-1}) - \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} (x(t_j) - x(t_{j-1})) \right|
\leq |x(t) - x(t_{j-1})| + \left| \frac{b(t_j) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} \right| |x(t_j) - x(t_{j-1})|
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$

Hence

$$
\lim_{n \to \infty} [x]_n(t) = x(t)
$$

uniformly on $[0,T]$. Since $F$ is continuous on $C_{a,b}[0,T]$,

$$
\lim_{n \to \infty} F([x]_n) = F(x).
$$

Let $F_n(x) = F([x]_n)$. Then for all $n \in \mathbb{N}$, $|F_n| = |F|$. So, by using the Bounded Convergence Theorem, we have the desired result. \hfill \Box

Lemma 3.2. Let $\varphi(t)$ be of bounded variation function on $[0,T]$ and let $x_0(t) = \int_0^t \varphi(s) db(s)$. Then $x_0 \in C_{a,b}[0,T]$ and we have

$$
\lim_{n \to \infty} \sum_{j=1}^{n} \frac{(x_0(t_j) - x_0(t_{j-1}))^2}{b(t_j) - b(t_{j-1})} = \int_0^T \varphi^2(s) db(s)
$$
Translation theorem on function space

\[
\lim_{n \to \infty} \sum_{j=1}^{n} \frac{((x(t_j) - a(t_j)) - (x(t_{j-1}) - a(t_{j-1}))) (x_0(t_j) - x_0(t_{j-1}))}{b(t_j) - b(t_{j-1})} = \int_0^T \varphi(s) dx(s) - \int_0^T \varphi(s) da(s).
\]

Proof. Since \( \varphi \in BV[0, T] \), \( x_0(t) = \int_0^t \varphi(s) db(s) \) is absolutely continuous on \([0, T]\). Observe that

\[
\sum_{j=1}^{n} \frac{(x_0(t_j) - x_0(t_{j-1}))^2}{b(t_j) - b(t_{j-1})} = \sum_{j=1}^{n} \left( \frac{x_0(t_j) - x_0(t_{j-1})}{b(t_j) - b(t_{j-1})} \right)^2 (b(t_j) - b(t_{j-1}))
\]

By using the Cauchy’s Mean Value Theorem in the above equation, we have

\[
\sum_{j=1}^{n} \frac{(x_0(t_j) - x_0(t_{j-1}))^2}{b(t_j) - b(t_{j-1})} = \sum_{j=1}^{n} \varphi^2(\xi_j) (b(t_j) - b(t_{j-1}))
\]

where \( \xi_j \in [t_{j-1}, t_j] \) for each \( j = 1, 2, \ldots, n \). Similarly, we have

\[
\sum_{j=1}^{n} \frac{((x(t_j) - a(t_j)) - (x(t_{j-1}) - a(t_{j-1}))) (x_0(t_j) - x_0(t_{j-1}))}{b(t_j) - b(t_{j-1})} = \sum_{j=1}^{n} \left( \frac{x_0(t_j) - x_0(t_{j-1})}{b(t_j) - b(t_{j-1})} \right) (x(t_j) - x(t_{j-1}))
\]

\[
- \sum_{j=1}^{n} \left( \frac{x_0(t_j) - x_0(t_{j-1})}{b(t_j) - b(t_{j-1})} \right) (a(t_j) - a(t_{j-1}))
\]

\[
= \sum_{j=1}^{n} \varphi(\xi_j) (x(t_j) - x(t_{j-1})) - \sum_{j=1}^{n} \varphi(\xi_j) (a(t_j) - a(t_{j-1})).
\]
Hence equations (3.10) and (3.11) converge to the followings
\[
\int_0^T \varphi^2(s)dB(s) \quad \text{and} \quad \int_0^T \varphi(s)dx(s) - \int_0^T \varphi(s)da(s)
\]
as \(n \to \infty\), respectively. Thus we have the desired results. \(\square\)

**Theorem 3.3.** Let \(\varphi\) and \(x_0\) be given as in Lemma 3.2 and let \(F\) be a \(\mathcal{B}(C_{a,b}[0,T])\)-measurable functional. Then \(F(x + x_0)\) is \(\mathcal{B}(C_{a,b}[0,T])\)-measurable and

\[
(3.12) \quad \int_{C_{a,b}[0,T]} F(y) d\mu(y) = \int_{C_{a,b}[0,T]} F(x + x_0) J(x, x_0) d\mu(x)
\]
where

\[
(3.13) \quad J(x, x_0) = \exp\left\{-\frac{1}{2} \int_0^T \varphi^2(s)dB(s) + \int_0^T \varphi(s)da(s) - \int_0^T \varphi(s)dx(s)\right\}.
\]

**Proof.** It suffices to show the case in which the functional \(F\) is bounded on \(C_{a,b}[0,T]\). Let us first consider the case \(F\) is bounded, continuous, and \(F(y) = 0\) for any \(y \in \{x \in C_{a,b}[0,T] : ||x|| > M\}\) where \(M > 0\). Then \(F(x + x_0)\) is measurable and there exists a positive real number \(K\) such that \(|F(x)| \leq K\) for all \(x \in C_{a,b}[0,T]\).

Let \(\tau : 0 = t_0 < t_1 < \cdots < t_n = T\) be a partition of \([0,T]\). Define a function \(G\) on \(\mathbb{R}^n\) by \(G(\vec{\xi}) = F(\vec{\xi})\) for each \(\vec{\xi} \in \mathbb{R}^n\). Then \(G\) is bounded and continuous. Hence we see that

\[
(3.14) \quad F([y]_n) = G(y(t_1), \cdots, y(t_n)).
\]

By using (3.14) and the Change of Variables Theorem, we have

\[
(3.15) \quad \int_{C_{a,b}[0,T]} F([y]_n) d\mu(y)
\]
\[
= \int_{C_{a,b}[0,T]} G(y(t_1), \cdots, y(t_n)) d\mu(y)
\]
\[
= \int_{\mathbb{R}^n} \left(\frac{1}{2\pi} \prod_{j=1}^n (b(t_j) - b(t_{j-1}))\right)^{-1/2} G(v_1, \cdots, v_n)
\]
\[
\cdot \exp\left\{-\frac{1}{2} \sum_{j=1}^n \left(\frac{(v_j - a(t_j)) - (v_{j-1} - a(t_{j-1}))}{b(t_j) - b(t_{j-1})}\right)^2\right\} dv.
\]
Let $\beta_j = x_0(t_j)$ and $u_j = v_j - \beta_j$. Then we see that
\begin{equation}
F([x + x_0]_n) = G(x(t_1) + x_0(t_1), \ldots, x(t_n) + x_0(t_n))
= G(x(t_1) + \beta_1, \ldots, x(t_n) + \beta_n))
\tag{3.16}
\end{equation}
and
\begin{equation}
((v_j - a(t_j)) - (v_{j-1} - a(t_{j-1})))^2
= ((u_j - a(t_j)) - (u_{j-1} - a(t_{j-1})))^2 + (\beta_j - \beta_{j-1})^2
+ 2(\beta_j - \beta_{j-1})(u_j - a(t_j)) - (u_{j-1} - a(t_{j-1})))
\tag{3.17}
\end{equation}
By applying (3.16) and (3.17) above to the last equation of (3.15), we have
\begin{equation}
\int_{C_{a,b}[0,T]} F([y]_n) d\mu(y)
= \exp \left\{- \frac{1}{2} \sum_{j=1}^{n} \frac{(x_0(t_j) - x_0(t_{j-1}))^2}{b(t_j) - b(t_{j-1})} \right\} \int_{C_{a,b}[0,T]} F([x + x_0]_n)
\cdot \exp \left\{- \sum_{j=1}^{n} \frac{(x(t_j) - a(t_j)) - (x(t_{j-1}) - a(t_{j-1})))}{b(t_j) - b(t_{j-1})}
\cdot (x_0(t_j) - x_0(t_{j-1})) \right\} d\mu(x).
\tag{3.18}
\end{equation}
Assume that $x + x_0 \notin \{ x \in C_{a,b}[0,T] : \|y\| > M \}$. Then we see that
\[\| [x + x_0]_n \| \leq \| x + x_0 \| \leq M \]
and so we have
\begin{equation}
|x(t_j)| \leq |x_0(t_j)| + |x(t_j) + x_0(t_j)| \leq \| x_0 \| + M
\tag{3.19}
\end{equation}
for any $x \in C_{a,b}[0,T]$. By using (3.11) and (3.19), we obtain that

$$\left| \sum_{j=1}^{n} \frac{((x(t_j) - a(t_j)) - (x(t_{j-1}) - a(t_{j-1}))) (x_0(t_j) - x_0(t_{j-1}))}{b(t_j) - b(t_{j-1})} \right|$$

$$= \left| \sum_{j=1}^{n} \varphi(\xi_j)(x(t_j) - x(t_{j-1})) - \sum_{j=1}^{n} \varphi(\xi_j)(a(t_j) - a(t_{j-1})) \right|$$

$$\leq \left| \sum_{j=1}^{n} \varphi(\xi_j)(x(t_j) - x(t_{j-1})) \right| + \left| \sum_{j=1}^{n} \varphi(\xi_j)(a(t_j) - a(t_{j-1})) \right|$$

$$= \left| \varphi(\xi_n)x(t_n) - \sum_{j=1}^{n} (\varphi(\xi_j) - \varphi(\xi_{j-1})) x(t_{j-1}) \right|$$

$$+ \left| \varphi(\xi_n)a(t_n) - \sum_{j=1}^{n} (\varphi(\xi_j) - \varphi(\xi_{j-1})) a(t_{j-1}) \right|$$

$$\leq |\varphi(\xi_n)||x(t_n)| + \sum_{j=1}^{n} |\varphi(\xi_j) - \varphi(\xi_{j-1})||x(t_{j-1})|$$

$$+ |\varphi(\xi_n)||a(t_n)| + \sum_{j=1}^{n} |\varphi(\xi_j) - \varphi(\xi_{j-1})||a(t_{j-1})|$$

$$\leq (\|x_0\| + M + \|a\||)(\|\varphi\| + V_0^T(\varphi))$$

where $V_0^T(\varphi)$ is the total variation of $\varphi$ on $[0,T]$. So the integrand of (3.18) is bounded by the following

$$K \exp\{([\|x_0\| + M + \|a\||)(\|\varphi\| + V_0^T(\varphi))\}.$$

Thus, by using the Bounded Convergence Theorem, (3.8), and (3.9), the expression (3.17) converges to

$$\exp\left\{-\frac{1}{2} \int_0^T \varphi^2(s) db(s) \right\}$$

$$\cdot \int_{C_{a,b}[0,T]} F(x + x_0) \exp\left\{- \int_0^T \varphi(s) dx(s) + \int_0^T \varphi(s) da(s) \right\} d\mu(x).$$

Hence by using Lemma 2.1, we obtain the equation (3.12) above.
Now, let $F$ be a nonnegative, bounded and continuous functional on $C_{a,b}[0,T]$. For each $n \in \mathbb{N}$, define the function $M_n$ by

$$M_n(u) = \begin{cases} 
1 & (0 \leq u \leq n) \\
n + 1 - u & (n \leq u \leq n + 1) \\
0 & (n + 1 \leq u). 
\end{cases}$$

Then $M_n$ is a continuous real valued function. Let $F_n(x) = F(x)M_n(\|x\|)$. Then $F_n$ satisfies the hypothesis of the first case. So proceeding as in the proof above, we obtain

$$\int_{C_{a,b}[0,T]} F_n(y)\,d\mu(y) = \exp\left\{-\frac{1}{2} \int_0^T \varphi^2(s)\,db(s) + \int_0^T \varphi(s)\,da(s)\right\} \cdot \int_{C_{a,b}[0,T]} F_n(x + x_0) \exp\left\{-\int_0^T \varphi(s)\,dx(s)\right\} \,d\mu(x).$$

Since $\{F_n\}$ is a monotone increasing sequence of functionals, $F_n \to F$ as $n \to \infty$. Hence by using the Monotone Convergence Theorem, we have the desired result. \qed

**Theorem 3.4.** Let $\varphi$, $x_0$, and $F$ be given as in Theorem 3.3. Then

$$\int_{C_{a,b}[0,T]} F(x + x_0)\,d\mu(x) = \exp\left\{-\frac{1}{2} \int_0^T \varphi^2(s)\,db(s) - \int_0^T \varphi(s)\,da(s)\right\} \cdot \int_{C_{a,b}[0,T]} F(x) \exp\left\{\int_0^T \varphi(s)\,dx(s)\right\} \,d\mu(x).$$

**Proof.** Let $G(x) = F(x)\exp\{\int_0^T \varphi(s)\,dx(s)\}$. Then by using equa-
tion (2.12), we have
\[
\int_{C_{a,b}[0,T]} F(x) \exp \left\{ \int_0^T \varphi(s)dx(s) \right\} d\mu(x) \\
= \int_{C_{a,b}[0,T]} G(x) d\mu(x) \\
= \int_{C_{a,b}[0,T]} G(x + x_0) J(x, x_0) d\mu(x) \\
= \int_{C_{a,b}[0,T]} F(x + x_0) \exp \left\{ \int_0^T \varphi(s)dx(s) + x_0(s) \right\} d\mu(x) \\
\cdot \exp \left\{ -\frac{1}{2} \int_0^T \varphi^2(s)db(s) + \int_0^T \varphi(s)da(s) - \int_0^T \varphi(s)dx(s) \right\} d\mu(x) \\
= \exp \left\{ \frac{1}{2} \int_0^T \varphi^2(s)db(s) + \int_0^T \varphi(s)da(s) \right\} \int_{C_{a,b}[0,T]} F(x + x_0) d\mu(x).
\]
Hence we have the desired result. \hfill \Box

4. The general translation theorem

In this section we consider the general translation theorem for functionals on $C_{a,b}[0,T]$.

For $u, v \in L^2_{a,b}[0,T]$, let
\[
(u, v)_{a,b} = \int_0^T u(t)v(t) d[b(t)] + |a|[t].
\]
Then $(\cdot, \cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u, u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space.

Let $\{e_j\}_{j=1}^\infty$ be a complete orthogonal set of real-valued functions of bounded variation on $[0,T]$ such that
\[
(e_j, e_k)_{a,b} = \begin{cases} 
0, & j \neq k \\
1, & j = k
\end{cases}.
\]
and for each \( v \in L^2_{a,b}[0,T] \), let
\[
v_n(t) = \sum_{j=1}^{n} (v, e_j)_{a,b} e_j(t)
\]
for \( n = 1, 2, \cdots \). Then for each \( v \in L^2_{a,b}[0,T] \), the Paley-Wiener-
Zygmund (PWZ) stochastic integral \( \langle v, x \rangle \) is defined by the formula
\[
\langle v, x \rangle = \lim_{n \to \infty} \int_0^T v_n(t) dx(t)
\]
for all \( x \in C_{a,b}[0,T] \) for which the limit exists.

**Remark 4.1.** Following are some facts about the PWZ stochastic integral
(i) For each \( v \in L^2_{a,b}[0,T] \), the PWZ integral \( \langle v, x \rangle \) exists for \( \mu \)-a.e. \( x \in C_{a,b}[0,T] \).
(ii) The PWZ integral \( \langle v, x \rangle \) is essentially independent of the complete
orthonormal set \( \{e_j\}_{j=1}^{\infty} \).
(iii) If \( v \in BV[0,T] \), then PWZ integral \( \langle v, x \rangle \) equals the Riemann
-Stieltjes integral \( \int_0^T v(s) dx(s) \) for \( s \)-a.e. \( x \in C_{a,b}[0,T] \).
(iv) The PWZ integral has the expected linearity properties.

**Lemma 4.1.** Let \( \varphi_n \in C_{a,b}[0,T] \cap BV[0,T] \) for each \( n \in \mathbb{N} \) and let \( \varphi_n \) converge in the space \( L^2_{a,b}[0,T] \) as \( n \to \infty \). Then for any real number \( \lambda \), \( \exp\{\lambda \int_0^T \varphi_n(t) dx(t)\} \) converges in the space \( L^2(C_{a,b}[0,T]) \) as \( n \to \infty \).

**Proof.** The proof given in [3] with the current hypotheses on \( a(t) \) and \( b(t) \) also works here. \( \square \)

Now, we obtain the general translation theorem of a functional on \( C_{a,b}[0,T] \).

**Theorem 4.2.** Let \( \varphi(t) \in L^2_{a,b}[0,T] \) and let \( x_0(t) = \int_0^t \varphi(s) db(s) \). If \( F \) be a \( B(C_{a,b}[0,T]) \)-measurable functional on \( C_{a,b}[0,T] \), then \( F(x+x_0) \) is \( B(C_{a,b}[0,T]) \)-measurable and
\[
\int_{C_{a,b}[0,T]} F(y) d\mu(y) = \exp\left\{-\frac{1}{2} \langle \varphi^2, b' \rangle + \langle \varphi, a' \rangle\right\} \cdot \int_{C_{a,b}[0,T]} F(x + x_0) \exp\{-\langle \varphi, x \rangle\} d\mu(x) \tag{4.1}
\]
where
\[(\varphi^2, b') = \int_0^T \varphi^2(t)b'(t)dt = \int_0^T \varphi(t)db(t)\]
and
\[(\varphi, a') = \int_0^T \varphi(t)a'(t)dt = \int_0^T \varphi(t)da(t).\]

Proof. It suffices to show the case in which the functional \(F\) is bounded and continuous on \(C_{a,b}[0, T]\). Let \(\{e_j\}_{j=1}^\infty\) be complete orthonormal set in \(L^2_{a,b}[0, T]\) with \(e_j \in C_{a,b}[0, T] \cap BV[0, T]\) for each \(j \in \mathbb{N}\). For each \(n \in \mathbb{N}\), let

\[(4.2) \quad \varphi_n(t) = \sum_{j=1}^n (\varphi, e_j)_{a,b} e_j(t).\]

Then \(\varphi_n \in C_{a,b}[0, T] \cap BV[0, T]\) and

\[(4.3) \quad \|\varphi_n - \varphi\|_{a,b} \to 0\]
as \(n \to \infty\). For each \(n \in \mathbb{N}\), define

\[(4.4) \quad x_{0,n}(t) = \int_0^t \varphi_n(s)db(s).\]

Then, by using equation (3.12), we see that

\[(4.5) \quad \int_{C_{a,b}[0, T]} F(y)d\mu(y) = \exp \left\{ - \frac{1}{2} \int_0^T \varphi_n^2(s)db(s) + \int_0^T \varphi_n(s)da(s) \right\} \cdot \int_{C_{a,b}[0, T]} F(x + x_{0,n}) \exp \left\{ - \int_0^T \varphi_n(s)dx(s) \right\} d\mu(x).\]
By using Cauchy-Schwarz inequality, we have
\begin{equation}
|x_0(t) - x_{0,n}(t)| = \left| \int_0^t (\varphi(s) - \varphi_n(s))db(s) \right|
\leq \int_0^T |(\varphi(s) - \varphi_n(s))\chi_{[0,t]}(s)|[b(s) + |a|(s)]
\leq \|\varphi - \varphi_n\|_{a,b} \sqrt{b(T)} + |a|(T).
\end{equation}
(4.6)

Hence by using (4.6) and (4.3), we obtain that
\begin{equation}
\|x_0 - x_{0,n}\| \rightarrow 0.
\end{equation}
(4.7)

Thus for all \(x \in C_{a,b}[0,T]\)
\begin{equation}
F(x + x_{0,n}) \rightarrow F(x + x_0).
\end{equation}
(4.8)

Since \(F\) is bounded, by applying the Bounded Convergence Theorem, we have
\begin{equation}
\int_{C_{a,b}[0,T]} |F(x + x_{0,n}) - F(x + x_0)|^2d\mu(x) \rightarrow 0.
\end{equation}
(4.9)

Note that
\begin{equation}
\int_0^T \varphi_n(t)dx(t) \rightarrow \langle \varphi, x \rangle \quad \mu - \text{a.e.} x \in C_{a,b}[0,T].
\end{equation}

So by using (4.3) and Lemma 4.1 with \(\lambda = -1\)
\begin{equation}
\exp \left\{ - \int_0^T \varphi_n(t)dx(t) \right\} \rightarrow \exp\{-\langle \varphi, x \rangle\}
\end{equation}
in the space \(L^2(C_{a,b}[0,T])\). Further, by equation (4.9) above, we obtain
\begin{equation}
\int_{C_{a,b}[0,T]} F(x + x_{0,n}) \exp \left\{ - \int_0^T \varphi_n(t)dx(t) \right\}d\mu(x)
\rightarrow \int_{C_{a,b}[0,T]} F(x + x_0) \exp\{-\langle \varphi, x \rangle\}d\mu(x).
\end{equation}
(4.10)

Hence by using equations (4.5) and (4.10) we have the desired equation (4.1). \(\square\)

We next use Theorem 4.2 to evaluate a translation theorem of functionals on \(C_{a,b}[0,T]\).
Theorem 4.3. Let \( \varphi, x_0, \) and \( F \) be given as in Theorem 4.2. Then

\[
\int_{C_{a,b}[0,T]} F(x + x_0) d\mu(x) = \exp\left\{-\frac{1}{2} \left( \varphi^2, b' \right) - (\varphi, a) \right\} \int_{C_{a,b}[0,T]} F(x) \exp\{\langle \varphi, x \rangle\} d\mu(x).
\]

Proof. Let \( G(x) = F(x) \exp\{\int_0^T \varphi(s) dx(s)\} \). Then, proceeding as in the proof of Theorem 3.4, we have the desired result. \( \square \)

References


Department of Mathematics
Dankook University
Cheonan 330-714, Korea
E-mail: jgchoi@dankook.ac.kr

Department of Mathematics
Dankook University
Cheonan 330-714, Korea
E-mail: yseo@dankook.ac.kr