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STABILITY OF F–HARMONIC MAPS

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Abstract. In this paper, we introduce the notion of F–harmonic maps and we study the stability of F–harmonic map.

1. Introduction

Harmonic maps are critical points of the energy functional defined on the space of smooth maps between Riemannian manifolds. We are interested in critical points of more general functional. Ara [1] introduced the notion of F–harmonic maps which unifies the F–harmonic maps and exponential harmonic maps. Let $F : [\,0,\infty) \to [\,0,\infty)$ be a C^2–function such that $F' > 0$ on $(0,\infty)$. For a smooth map $\phi : (M,g) \to (N,h)$ between Riemannian manifolds (M,g) and (N,h), we define the F–energy $E_F(\phi)$ of ϕ by

$$E_F(\phi) = \int_M F\left(\frac{|d\phi|^2}{2}\right) v_g,$$

where $|d\phi|$ denotes the Hilbert–Schmidt norm of the differential $d\phi \in \Gamma(T^*M \bigotimes \phi^{-1}TN)$ with respect to g and h and v_g is the volume element of (M,g). We call ϕ on F–harmonic map if it is a critical point of the F–energy functional. That is, ϕ is an F–harmonic map if and only if $\frac{d}{dt}E_F(\phi_t)\big|_{t=0} = 0$ for any compactly supported variation $\phi_t : M \to N$ $(-\epsilon < t < \epsilon)$ with $\phi_0 = \phi$.

Let ∇ and ${}^N\nabla$ denote the Levi–Civita connections of M and N, respectively. Let $\widetilde{\nabla}$ be the induced connection on $\phi^{-1}TN$ defined by $\widetilde{\nabla}_X W = {}^N\nabla_{\phi_* X}$, where X is a tangent vector of M and W is a section

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of $\phi^{-1}TN$. We choose a local orthogonal frame field $\{e_i\}_{i=1}^m$ on M. We define the F–tension field $\mathcal{T}_F(\phi)$ of ϕ by

$$\begin{aligned}\mathcal{T}_F(\phi) &= \sum_{i=1}^m \left[\widetilde{\nabla}_{e_i}\left\{F'\left(\frac{|d\phi|^2}{2}\right)\phi_* e_i\right\} - F'\left(\frac{|d\phi|^2}{2}\right)\phi_*\nabla_{e_i}e_i\right]\\ &= F'\left(\frac{|d\phi|^2}{2}\right)\tau(\phi) + \phi_*\left[grad\left\{F'\left(\frac{|d\phi|^2}{2}\right)\right\}\right],\end{aligned}$$

where $\tau(\phi) = \sum_{i=1}^m(\widetilde{\nabla}_{e_i}\phi_* e_i - \phi_*\nabla_{e_i}e_i)$ is the tension field of ϕ. Therefore a smooth map $\phi_t : M \to N$ is an F–harmonic map if and only if the F–tension field $\mathcal{T}_F = 0$. In this paper, we study the stability of F–harmonic maps.

2. Main results

Let $\phi : (M, g) \to (N, h)$ be a smooth map from m–dimensional Riemannian manifold (M, g) to a Riemannian manifold (N, h).

THEOREM 1. *[1] (The second variation formula) Let $\phi : M \to N$ be an F–harmonic map. Let $\phi_{s,t} : M \to N$ $(-\epsilon < s, t < \epsilon)$ be a compactly supported two–parameter variation such that $\phi_{0,0} = \phi$ and set $V = \frac{\partial\phi_{s,t}}{\partial t}\Big|_{s,t=0}$, $W = \frac{\partial\phi_{s,t}}{\partial s}\Big|_{s,t=0}$. Then*

$$\frac{\partial^2}{\partial s\partial t}E_F(\phi_{s,t})\Big|_{s,t=0} = \int_M F''\left(\frac{|d\phi|^2}{2}\right) < \widetilde{\nabla}V, d\phi >< \widetilde{\nabla}W, d\phi > v_g$$

$$+\int_M F'\left(\frac{|d\phi|^2}{2}\right)\left\{< \widetilde{\nabla}V, \widetilde{\nabla}W > - \sum_{i=1}^m h({}^N R(V, \phi_* e_i)\phi_* e_i, W)\right\} v_g$$

where $<,>$ is the inner product on $T^*M \bigotimes \phi^{-1}TN$ and ${}^N R$ is the curvature tensor of N. we put

$$I(V, W) = \frac{\partial^2}{\partial s\partial t}E_F(\phi_{s,t})\Big|_{s,t=0}$$

Definition. An F–harmonic map ϕ is called F–stable or stable if $I(V,W) \geq 0$ for any compactly supported vector field V along ϕ, or equivalently, the eigenvalues of the F–Jacobi operator $J_{F,\phi}$ are all nonnegative.

Theorem 2. *Let $\phi : M \to N$ be an F–harmonic map from a Riemannian manifold M to a Riemannian manifold N. Assume that $F'' \geq 0$ and N has nonpositive curvature. Then ϕ is stable.*

Proof. It follows from Theorem 1. □

Eells [4] showed that if $dim \geq 3$ for any homotopy class $\mathcal{H}$, then there exists a C^∞–Riemannian metric $\widetilde{g}$ on M conformal to g and C^∞–map ϕ in $\mathcal{H}$ such that $\phi : (M, \widetilde{g}) \to (N, h)$ is harmonic. Ara [1] showed that let $dim \geq 3$ and $F : [\,0, \infty) \to [\,0, \infty)$ be a smooth function such that $F' > 0$ on $[\,0, \infty)$ and $F''(0) = 0$. Then there is a smooth metric $\widetilde{g}$ on M conformally equivalent to g and a map $\phi \in \mathcal{H}$ such that $\phi : (M, \widetilde{g}) \to (N, h)$ is F–harmonic, where $\mathcal{H}$ is a homotopy class of a smooth given map $(M, \widetilde{g}) \to (N, h)$.

Theorem 3. *Let M be an m–dimensional compact Riemannian manifold and $F : [\,0, \infty) \to [\,0, \infty)$ a C^2–strictly increasing function such that*

$$mF''\left(\frac{m}{2}\right) + (2-m)F'\left(\frac{m}{2}\right) \geq 0$$

Then M is F–stable.

Proof.

$$\begin{aligned}
\int_M g(J_{F,id}(V), V)v_g &= F''\left(\tfrac{m}{2}\right)\int_M (divV)^2 v_g \\
&+F'\left(\frac{m}{2}\right)\int_M \sum_{i=1}^m \left\{g(\nabla_{e_i}V, \nabla_{e_i}V) - g(R^M(V,e_i)e_i, V)\right\} v_g \\
&= F''\left(\frac{m}{2}\right)\int_M (divV)^2 v_g + F'\left(\frac{m}{2}\right)\int_M g(J_{2,id}(V), V)v_g \\
&\geq \frac{1}{m}\left\{mF''\left(\frac{m}{2}\right) + (2-m)F'\left(\frac{m}{2}\right)\right\}\int_M (divV)^2 v_g \\
&\geq 0
\end{aligned}$$

Hence M is F–stable. □

THEOREM 4. *Let M be an m–dimensional compact Riemannian manifold which supports a nonisometric conformal vector field V and $F : [\,0,\infty) \to [\,0,\infty)$ a C^2–strictly increasing function. Then M is F–stable if and only if F satisfies*

$$mF''\left(\frac{m}{2}\right) + (2-m)F'\left(\frac{m}{2}\right) \geq 0$$

Proof. Since a vector field V on M is conformal if and only if

$$\mathcal{L}_V g = -\frac{2}{m}(divV)g, \quad \frac{1}{2}|\mathcal{L}_V g|^2 = \frac{2}{m}(divV)^2$$

where $\mathcal{L}_V g$ is the Lie derivative of the metric g. Then we have

$$\begin{aligned}
&\int_M g(J_{F,id}(V), V)v_g \\
&= F''\left(\frac{m}{2}\right)\int_M (divV)^2 v_g + F'\left(\frac{m}{2}\right)\int_M \left\{\frac{1}{2}|\mathcal{L}_V g|^2 - (divV)^2\right\} v_g \\
&= \frac{1}{m}\left\{mF''\left(\frac{m}{2}\right) + (2-m)F'\left(\frac{m}{2}\right)\right\}\int_M (divV)^2 v_g
\end{aligned}$$

If V is nonisometric conformal, we have $divV \not\equiv 0$ □

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