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CONVERGENCE TO FRACTIONAL BROWNIAN MOTION AND LOSS PROBABILITY

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Abstract. We study the weak convergence to Fractional Brownian motion and some examples with applications to traffic modeling. Finally, we get loss probability for queue-length distribution related to self-similar process.

1. Introduction

Traditional traffic models based on the Poisson process or, more generally, on short range dependent processes, cannot describe the behavior of actual LAN traffic. Because of tremendous burstiness of LAN traffic at any time scale, many researchers have studied long range dependent process and self-similar process.

Kelly ([4]) has considered the notion of effective bandwidth in the context of stochastic models for the statistical sharing of resources to figure out the loss probability. Chang and Zajic ([2]) apply the result on the effective bandwidth of stationary departure process to intree networks with time varying capacities and priority tandem queues. Recently, several researchers ([1],[4],[6],[10]) have proposed and developed the theory of effective bandwidth and loss probability as a tentative solution for various problems that arise in high speed digital networks, in particular ATM networks.

On the other hand, there has been a recent flood of literature and discussion on the tail behavior of queue-length distribution, motivated by potential applications to the design and control by high-speed telecommunication networks([3],[5]).

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In section 2, we define the effective bandwidth with a stationary source and introduce the effective bandwidth of Brownian motion and Fractional Brownian motion. In section 3, we study the weak convergence to Fractional Brownian motion and give some examples with applications to traffic modeling. In section 4, we obtain the loss probability, i.e. tail behavior of queue-length distribution, of self-similar process.

2. Definition and Preliminary

In this section we first define the effective bandwidth with a stationary source X_i which is the number of arrivals in the ith time unit.

DEFINITION 2.1. The effective bandwidth of $X(\tau) = \sum_{i=1}^{\tau} X_i$ is defined as

$$eb_X(\theta, \tau) = \frac{1}{\theta\tau} \log E[e^{\theta \sum_{i=1}^{\tau} X_i}], \quad 0 < \theta < \infty.$$

If X_i are independent, then

$$eb_X(\theta, \tau) = \sum_i eb_{X_i}(\theta, \tau).$$

Furthermore, for any fixed value of τ, $eb_X(\theta, \tau)$ is increasing in θ and

$$\frac{EX[0,\tau]}{\tau} \le eb_X(\theta, \tau) \le \frac{\bar{X}[0,\tau]}{\tau},$$

where $\bar{X}[0,\tau]$ is the essential supremum.

DEFINITION 2.2. A stochastic process $\{X(t)\}$ is said to be a Brownian motion if

1. $X(t)$ has stationary and independent increments
2. for $t > 0$, $X(t) \sim N(\mu, \sigma^2 t)$
3. $X(0) = 0$ $a.s.$

The effective bandwidth of a Browian motion is

$$eb(\theta, \tau) = \mu + \frac{\theta\sigma^2}{2},$$

where μ is the mean arrival rate and σ^2 is the variance of the arrival. A critical point of a Brownian Motion stream

$$\inf_{\tau \ge 0} \sup_{\theta \ge 0} \{\theta(B + C\tau) - \theta\tau(\mu + \frac{\theta\sigma^2}{2})\}$$

$$\tau^* = \frac{B}{C-\mu}, \theta^* = \frac{2(C-\mu)}{\sigma^2}.$$

Let $\rho_X(k)$ be the covariance of stationary stochastic process $X(t)$. Then we define the followings.

DEFINITION 2.3. A stationary stochastic process exhibits short range dependence if

$$\sum_{k=-\infty}^{\infty} |\rho_X(k)| < \infty$$

DEFINITION 2.4. A stationary stochastic process exhibits long range dependence if

$$\sum_{k=-\infty}^{\infty} |\rho_X(k)| = \infty$$

DEFINITION 2.5. A stochastic process $\{B_H(t)\}$ is said to be a Fractional Brownian motion(FBM) with Hurst parameter H if

1. $B_H(t)$ has stationary increments
2. for $t > 0$, $B_H(t)$ is normally distributed with mean 0
3. $B_H(0) = 0$ $a.s.$
4. The increments of $B_H(t)$, $Z(j) = B_H(j+1) - B_H(j)$ satisfy

$$\rho_Z(k) = \frac{1}{2}\{|k+1|^{2H} + |k-1|^{2H} - 2k^{2H}\}$$

A standard example of a long range dependent process is fractional Brownian motion, Hurst parameter $H > 1/2$. If $H < 1/2$, then this fractional Brownian motion exhibits short range dependence. On the other hand, the effective bandwidth of a FBM is

$$eb(\theta, \tau) = \mu + \frac{\theta\sigma^2}{2}\tau^{2H-1},$$

and the critical points are

$$\tau^* = \frac{B}{C-\mu}\frac{H}{1-H}, \quad \theta^* = \frac{B + (C-\mu)\tau^*}{\sigma^2(\tau^*)^{2H}}.$$

DEFINITION 2.6. A continuous process $X(t)$ is self-similar with self-similarity parameter $H \geq 0$ if it satisfies the condition:

$$X(t) \stackrel{d}{=} c^{-H}X(ct), \quad \forall\, t \geq 0, \forall c > 0,$$

where the equality is in the sense of finite-dimensional distributions.

Brownian motion and Fractional Brownian motion are two important examples of self-similar process.

3. Convergence to Fractional Brownian motion

Let $Y^i(j)$ be the number of arrivals in the jth time unit of ith source. Let

$$Y_M(j) = \sum_{i=1}^{M} (Y^i(j) - E(Y^i(j))),$$

and $\tau(k)$ denote the covariance of $Y_1(j)$.

LEMMA 3.1. [8] *The stationary sequence*

$$\frac{1}{M^{1/2}} Y_M(j)$$

converges in the sense of finite dimensional distributions to $G_H(j)$*, where* $G_H(j)$ *represents a stationary Gaussian process with covariance function of the same form as* $\tau(k)$*, as* $M \to \infty$*.*

THEOREM 3.1.

$$\lim_{T\to\infty} \lim_{M\to\infty} \frac{1}{T^H M^{1/2}} \sum_{j=0}^{[Tt]} Y_M(j)$$

converges in the sense of finite dimensional distributions to $\{\sigma_0 B_H(t) | 0 \le t \le 1\}$*.*

Furthermore, as $M \to \infty$ *and* $T \to \infty$*,*
(a) *(Long Range dependence) If*

$$\rho(k) \sim ck^{2H-2}, \;\; c > 0 \;\; \text{and} \;\; 1/2 < H < 1,$$

then $\sigma_0^2 = \dfrac{c}{H(2H-1)}$*.*
(b) *If*

$$\sum_{k=1}^{\infty} |\rho(k)| < \infty \;\; \text{and} \;\; \sum_{k=1}^{\infty} \rho(k) = c > 0,$$

then $\sigma_0^2 = c$*.*
(c) *(Short Range dependence)*

$$\rho(k) \sim ck^{2H-2}, \;\; c < 0 \;\; \text{and} \;\; 0 < H < 1/2,$$

then $\sigma_0^2 = -\dfrac{c}{H(2H-1)}$.

Proof. Set $Z_j = 1/M^{1/2} Y_M(j)$. By Lemma 3.1, Z_j converges in the sense of finite dimensional distributions to $G_H(j)$ as M goes to infinity. By Theorem 7.2.11 of [9], the finite dimensional distributions of $N^{-H} \sum_{j=1}^{[Nt]} G_H(j)$ converges to those of $\{\sigma_0 B_H(t), 0 \leq t \leq 1\}$. □

THEOREM 3.2. *Let X_t be the autoregressive process of order one, i.e. $X_t = \phi_1 X_{t-1} + a_t$, where $a_t \sim N(0,1)$ for each t. Then*

$$\lim_{T\to\infty} \lim_{M\to\infty} \sum_{j=0}^{[Tt]} Y_M(j) = \frac{\phi_1}{1-\phi_1} B(t).$$

Proof.

$$(1 - \phi_1 B) X_t = a_t,$$

i.e.

$$X_t = \sum_{j=0}^{\infty} \phi_1^j a_{t-j}.$$

$$\mathrm{Cov}_{X_t}(k) = \phi_1^k, \ k \geq 1, |\phi_1| < 1.$$

Therefore,

$$\rho(k) = \phi_1^k,$$

for large M. Since

$$\sum \rho(k) = \sum \phi_1^k = \frac{\phi_1}{1-\phi_1} < \infty.$$

Then, from theorem 3.1, we get

$$\lim_{T\to\infty} \lim_{M\to\infty} \sum_{j=0}^{[Tt]} Y_M(j) = \frac{\phi_1}{1-\phi_1} B_{1/2}(t) = \frac{\phi_1}{1-\phi_1} B(t).$$

□

EXAMPLE 3.1 (FARIMA(p,d,q)). Let $Y^i(j) = b_i(-d) a_{j-i}$. Then

$$\rho(k) \sim c k^{2d-1} \ \ as \ \ k \to \infty$$

where $H = d + 1/2, -1/2 < d < 1/2$ and $c = \dfrac{\Gamma(1-2d)\sin(\pi d)}{\pi}$.

By Theorem 3.1,

$$\lim_{T\to\infty}\lim_{M\to\infty}\frac{1}{T^H M^{1/2}}\sum_{j=0}^{[Tt]}\sum_{i=1}^{M}(Y^i(j)) = \sqrt{\frac{c}{H(2H-1)}}B_H(t).$$

EXAMPLE 3.2 (Binary sequence). Let $Y^i(j)$ denote the increment process for the ith stationary binary sequence $W^i(t)$ that it generates, where $W^i(t) = 1$ means that there is a packet at time t and $W^i(t) = 0$ means that there is no packet.
We get

$$\rho(k) \sim ck^{2H-2},$$

as $k \to \infty$ and $E[Y^i(j)] = \frac{\mu_1}{\mu_1+\mu_2}$ if $E[\text{Onperiod}] = \mu_1$ and $E[\text{Offperiod}] = \mu_2$. By Theorem 3.1,

$$\lim_{T\to\infty}\lim_{M\to\infty}\frac{1}{T^H M^{1/2}}\left\{\sum_{j=0}^{[Tt]}\sum_{i=1}^{M}(Y^i(j)) - \sum_{i=1}^{M}\frac{\mu_1}{\mu_1+\mu_2}t\right\}$$
$$= \sqrt{\frac{c}{H(2H-1)}}B_H(t).$$

4. Loss Probability of Stochastic Process

Let A_τ be the amount of work that arrives to be processed in $[0,\tau]$ and S_τ be the amount of work that can be processed in the same time interval. Then the workload process is

$$Q_\tau = A_\tau - S_\tau.$$

and queue-length is defined

$$Q = sup_\tau Q_\tau.$$

THEOREM 4.1. [2]

$$\lim_{b\to\infty} logP(Q > b) = -\delta,$$

where

$$\delta = \sup\{\theta : \lambda(\theta) \le 0\}$$

and

$$\lambda(\theta) = \lim_{\tau\to\infty}\frac{1}{\tau}\log Ee^{\theta Q_\tau}.$$

Note that for long range dependent data,

$$logP(Q > b) \sim -\delta b^{\gamma},$$

where $\gamma = 2(1 - H)$.

THEOREM 4.2 ([7], Prop. 9).

$$\limsup_{N\to\infty} \frac{1}{N} \log P(Q > Nb) \le -\{\theta^*(b + c\tau^*) - \theta^*\tau^* eb(\theta^*, \tau^*)\}.$$

From now on, we study the property and loss probability of self-similar process. Self-similar processes are of interest in probability theory because they are connected with limit theorems. Namely, every limit process with scaling is self-similar as the following lemma states.

THEOREM 4.3 ([9]). *Suppose $X(t)$ is continuous in probability of $t = 0$ and the distribution of $X(t)$ is nondegenerate for each $t > 0$. If there exist a stochastic process $Y(t)$ and real $\{K(T); T \ge 0\}$ with $K(T) > 0, \lim_{T\to\infty} K(T) = \infty$ such that as $T \to \infty$,*

$$\frac{1}{K(T)} Y(Tt) \Rightarrow X(t),$$

where $\Rightarrow$ means the convergence of finite-dimensional distributions, then for some $H > 0$, $X(t)$ is self-similar process.

Furthermore, $K(T)$ is of the form $K(T) = T^H L(T)$, where $L(T)$ is a slowly varying function.

Let $A_\tau = \mu\tau + X_H(\tau)$, where $X_H(\tau)$ is a self-similar process.

THEOREM 4.4. *For any $a > 0$,*

$$A_{a\tau} = c_{a,H,\mu}(\tau) + a^H A_\tau,$$

where $c_{a,H,\mu} = \mu\tau(a - a^H)$.

Proof.

$$\begin{aligned} A_{a\tau} &= \mu a\tau + X_H(a\tau) \\ &= \mu\tau(a - a^H) + a^H(\mu\tau + X_H(\tau)) \\ &= \mu\tau(a - a^H) + a^H A_\tau. \end{aligned}$$

□

Let c be a service rate and Q_τ be a waiting length at τ. Then

$$Q_\tau = A_\tau - c\tau$$

and queueing length

$$Q = \sup_\tau Q_\tau$$

is defined.

THEOREM 4.5. *For any $b > 0$,*

$$P(Q > b) > \sup_\tau P\left(X_H(1) > \frac{b - (\mu - c)\tau}{\tau^H}\right).$$

Proof.

$$\begin{aligned} P(\sup_\tau((A(\tau) - c\tau) > b) &= P(\sup_\tau(X_H(\tau) + \mu\tau - c\tau > b) \\ &> \sup_\tau P(X_H(\tau) + (\mu - c)\tau > b) \\ &= \sup_\tau P(X_H(\tau) > b - (\mu - c)\tau) \\ &= \sup_\tau P\left(X_H(1) > \frac{b - (\mu - c)\tau}{\tau^H}\right). \end{aligned}$$

□

If $X_H(1) \sim S_\alpha(\sigma, \beta, \mu)$ with $0 < \alpha < 2$, then left hand side of Theorem 4.5 equals

$$C_\alpha \frac{1 + \beta}{2} \sigma^\alpha \left(\frac{b - (\mu - c)\tau}{\tau^H}\right)^\alpha,$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx\right)^{-1}.$$

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